View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF DIFFERENTIAL EQUATIONS 99, 306–341 (1992)

C^p Singularity Theory and Heteroclinic Bifurcation with a Distinguished Parameter

STEPHEN SCHECTER*

Mathematics Department, North Carolina State University, Raleigh, North Carolina 27695

Received September 25, 1990; revised January 7, 1991

1. INTRODUCTION

Let

$$\dot{z} = h(z, \lambda) \tag{1.1}$$

be a C^∞ vector field in the plane depending on a distinguished real parameter λ. We suppose that for $\lambda = 0$ there are distinct equilibria at p_0 and q_0 and a heteroclinic orbit from p_0 to q_0. We concentrate on the case in which p_0 is a hyperbolic saddle, q_0 has one negative and one zero eigenvalue, and the heteroclinic orbit lies in the unique invariant curve through q_0 that is tangent to the eigendirection for the negative eigenvalue. As λ varies, the heteroclinic orbit may break and the semihyperbolic equilibrium may bifurcate. These bifurcations are described by a pair of functions $(f_1(x, \lambda), f_2(x, \lambda))$, where x is a coordinate along the center subspace at q_0; since center mainfolds are not in general C^∞, it turns out that at least one of these functions need not be C^∞.

The goal of this paper is to show how to use singularity theory to find normal forms, recognition criteria, and universal unfoldings for the pair (f_1, f_2). We then study some of the normal forms of low codimension.

The singularity theory we need is C^p, not C^∞. We have therefore included in this paper an exposition of C^p singularity theory with a distinguished parameter. In particular, our treatment of the C^p recognition problem with a distinguished parameter seems to be new and should be useful in other problems.

In [7] we studied an infinite codimension problem of the type considered here. Since the problem was of infinite codimension, there was no normal form and no universal unfolding.

* Research supported by National Science Foundation Grant DMS 9002803.

0022-0396/92 $5.00
Copyright © 1992 by Academic Press, Inc.
All rights of reproduction in any form reserved.

As with [7], the present work is motivated by the study of shock solutions of conservation laws. A *system of two conservation laws* in one space-dimension is a partial differential equation of the form

$$U_t + F(U)_x = 0, \tag{1.2}$$

where $U \in \mathbb{R}^2$ and $F\colon \mathbb{R}^2 \to \mathbb{R}^2$. A *shock solution* with speed s of (1.2) is a discontinuous function

$$U = \begin{cases} U_- & x < st \\ U_+ & x > st \end{cases} \tag{1.3}$$

that satisfies the *Rankine–Hugoniot condition*

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0.$$

The shock (1.3) has a *viscous profile* if the equation

$$U_t + F(U)_x = \varepsilon U_{xx} \tag{1.4}$$

has a traveling wave solution

$$U\left(\frac{x - st}{\varepsilon}\right) \tag{1.5}$$

with

$$\lim_{t \to \pm\infty} U(\xi) = U_\pm, \qquad \lim_{t \to \pm\infty} U'(\xi) = 0. \tag{1.6}$$

If we substitute (1.5) into (1.4), integrate once, and use the left-hand boundary conditions from (1.6), we find that a shock solution of (1.2) with left state U_-, speed s, and viscous profile corresponds to a heteroclinic orbit of

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{1.7}$$

from U_- to a second equilibrium U_+. Equation (1.7) is a vector field on $\mathbb{R}^2$ with distinguished parameter s and unfolding parameters U_-.

Suppose, for example, that for some fixed U_-, as s varies, the phase portrait of (1.7) undergoes the changes portrayed in Fig. 1.1. The fact that the separatrix crossing bifurcation ($s = s_1$) occurs before the saddle-node bifurcation ($s = s_2$) prevents the existence of a heteroclinic orbit from U_- to the node U_+ that appears by saddle-node bifurcation at $s = s_2$. This example indicates the relevance of heteroclinic bifurcation diagrams to the study of shocks with viscous profile.

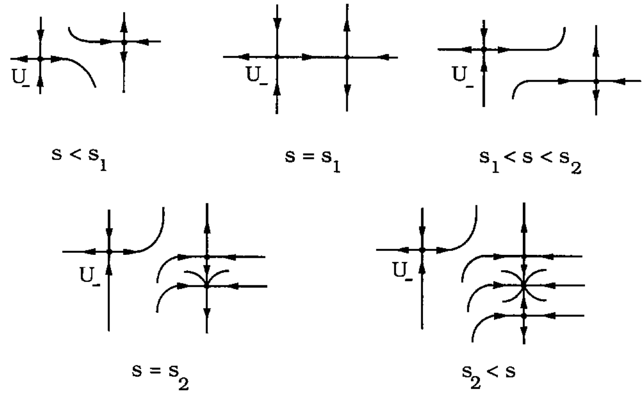

FIGURE 1.1

As Fig. 1.1 indicates, generic one-parameter vector fields in $\mathbb{R}^2$ exhibit (among other bifurcations) separatrix crossings and saddle-node bifurcations of equilibria; moreover, at each value of the parameter they exhibit just one of these bifurcations. By contrast, the one-parameter families studied in the present paper exhibit heteroclinic bifurcation and equilibrium degeneracy at the same time. An understanding of how these families unfold should be helpful in the study of generic properties of conservation laws. Furthermore, recognition criteria for degenerate one-parameter families should be useful in dealing with concrete problems.

The application of singularity theory to heteroclinic bifurcation problems involving semihyperbolic equilibria was pioneered by Vegter [9], who considered problems without a distinguished parameter. We shall adapt Vegter's approach to problems with a distinguished parameter, much as Golubitsky and Schaeffer [3] adapted the singularity theory of Mather to equilibrium bifurcation.

We begin by recalling some terminology. Suppose q is an equilibrium of a differential equation $\dot{z} = h(z)$, $z \in \mathbb{R}^2$, and the eigenvalues of $Dh(q)$ are μ_1 and μ_2. The equilibrium q is *hyperbolic* if $\mathrm{Re}(\mu_i) \neq 0$, $i = 1, 2$. If both μ_i are real, q is a *saddle* if $\mu_1\mu_2 < 0$ and a *node* if $\mu_1\mu_2 > 0$. If exactly one $\mu_i = 0$, q is *semihyperbolic*.

Suppose μ_i is an eigenvalue of a saddle or the nonzero eigenvalue of a semihyperbolic equilibrium, and v_i is a corresponding eigenvector of $Dh(q)$. There is a unique invariant curve through q tangent to v_i, called the *stable* (resp. *unstable*) *manifold* of q if $\mu_i < 0$ (resp. $\mu_i > 0$). Suppose q is a node with eigenvalues $\mu_1 < \mu_2 < 0$ (resp. $\mu_1 > \mu_2 > 0$) and corresponding eigenvectors v_1 and v_2. There is a unique invariant curve through q tangent to v_1, called the *strong stable* (resp. *strong unstable*) manifold of q. It consists of two orbits; all other nearby orbits approach q along v_2. Finally, suppose

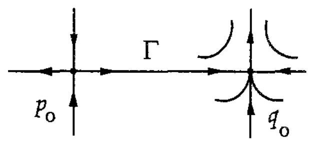

FIGURE 1.2

q is a semihyperbolic equilibrium and v is an eigenvector of $Df(q)$ for the eigenvalue 0. There is an invariant curve through q tangent to v called the *center manifold* of q; it need not be unique.

We now explain how a heteroclinic bifurcation problem (1.1) involving a semihyperbolic equilibrium can be reduced to a problem in C^p singularity theory. We basically follow [9] but add a distinguished parameter. We assume that (1.1) has, for $\lambda = 0$, a hyperbolic saddle p_0, a semihyperbolic equilibrium q_0, and a heteroclinic orbit Γ that lies in both the unstable manifold of p_0 and the stable manifold of q_0; see Fig. 1.2. In a neighborhood $V \times (-\varepsilon, \varepsilon)$ of $(q_0, 0)$ there are new C^{p+1} coordinates $y = \phi(z, \lambda)$, p arbitrarily large but finite, such that in the new coordinates, $(q_0, 0)$ is the origin, the plane $y_1 = 0$ is invariant (it is the *extended center manifold* of the origin), and the family of lines $(y_2, \lambda) = \text{constant}$ is mapped into itself by the flow. Thus $\dot{z} = h(z, \lambda)$ becomes

$$\begin{aligned} \dot{y}_1 &= y_1 I(y_1, y_2, \lambda), \\ \dot{y}_2 &= J(y_2, \lambda), \end{aligned} \tag{1.8}$$

where $y_1 I$ and J are only C^p. Let Σ be a line segment transverse to Γ in V. Near p_0 there is a nearby hyperbolic saddle $p(\lambda)$ for each λ near 0. The branch of the unstable manifold of $p(\lambda)$ near Γ meets Σ in a point with y_2-coordinate $y_2(\lambda)$; see Fig. 1.3. We now define a C^p function $g = (g_1, g_2)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ by

$$\begin{aligned} g_1(y_2, \lambda) &= y_2(\lambda) - y_2, \\ g_2(y_2, \lambda) &= J(y_2, \lambda). \end{aligned}$$

The function g determines the flow of $\dot{z} = h(z, \lambda)$ in a fixed neighborhood of Γ. For example, an equilibrium near q_0 corresponds to a zero of g_2, and

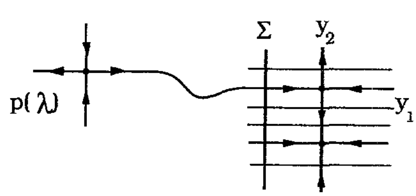

FIGURE 1.3

a heteroclinic orbit from $p(\lambda)$ to a hyperbolic saddle near q_0 corresponds to a solution of $g_1 = g_2 = 0$ at which $\partial g_2/\partial y_2 > 0$. The flow illustrated in Fig. 1.3 has such a common zero of g_1 and g_2.

Let us make three remarks:

1. A solution of $g_1 = g_2 = 0$ at which $\partial g_2/\partial y_2 < 0$ corresponds to a heteroclinic orbit that lies in the strong stable manifold of a node near q_0. We shall refer to any solution of $g_1 = g_2 = 0$ as a *heteroclinic bifurcation.*

2. Since p_0 is a hyperbolic saddle, we have arranged that $(\partial g_1/\partial y_2)(0, 0) < 0$ (corresponding to the sign of the eigenvalue of $Dh(p_0)$ for the eigendirection transverse to Γ). If p_0 and q_0 are both semihyperbolic, then both g_1 and g_2 have zero partial derivative with respect to the spatial variable; see [9] for the construction of g_1 and g_2 in this case.

3. If the differential equation also depends on unfolding parameters α, i.e., if we consider $\dot{z} = H(z, \lambda, \alpha)$ with $H(z, \lambda, 0) = h(z, \lambda)$, then the functions g_i extend to functions $G_i(y_2, \lambda, \alpha)$ with $G_i(y_2, \lambda, 0) = g_i(y_2, \lambda)$.

The appropriate equivalence relation for this situation is the following. Let $\mathscr{D}$ be the group of diagonal 2×2 matrices with positive diagonal entries. We shall say that (g_1, g_2) is $\mathscr{D}$ equivalent to (f_1, f_2) provided

$$\begin{pmatrix} g_1(y_2, \lambda) \\ g_2(y_2, \lambda) \end{pmatrix} = \begin{pmatrix} A(y_2, \lambda) & 0 \\ 0 & C(y_2, \lambda) \end{pmatrix} \begin{pmatrix} f_1(Y(y_2, \lambda), \Lambda(\lambda)) \\ f_2(Y(y_2, \lambda), \Lambda(\lambda)) \end{pmatrix},$$

where

$$A > 0, C > 0, Y(0, 0) = \Lambda(0) = 0, \frac{\partial Y}{\partial y_2} > 0, \frac{d\Lambda}{d\lambda} > 0. \tag{1.9}$$

As in [3] the change of variables in λ is independent of y_2. The functions A, C, Y, Λ are of class C^s for some s, $1 \leqslant s \leqslant p$. $\mathscr{D}$-equivalence takes the zero sets of g_1 and g_2 to those of f_1 and f_2, respectively; moreover, if $g_i(y_2, \lambda) = 0$, then $(\partial g_i/\partial y_2)(y_2, \lambda)$ and $(\partial f_i/\partial y_2)(Y(y_2, \lambda), \Lambda(\lambda))$ have the same sign. The more familiar relation of v-equivalence or contact-equivalence for mappings into $\mathbb{R}^2$ allows multiplication by an arbitrary invertible 2×2 matrix [3]. It is appropriate when one is interested only in the intersection of the zero sets of g_1 and g_2.

Since we assume that just one of p_0 and q_0 is semihyperbolic, an alternative to Vegter's approach that uses the *separation function* is available and is easier to use in concrete problems. This alternative was described in [5] for problems without a distinguished parameter; we now describe it for problems with a distinguished parameter.

Again suppose that $\dot{z} = h(z, \lambda)$ has, for $\lambda = 0$, a hyperbolic saddle at p_0 connected by a heteroclinic orbit Γ to a semihyperbolic equilibrium q_0 as before. We make an affine change of coordinates on z-space so that q_0 corresponds to $y=0$, and the stable (resp. center) eigenspace of $D_z h(q_0, 0)$ corresponds to the line $y_2=0$ (resp. $y_1=0$). Then the extended center manifold of $\dot{z}=h(z, \lambda)$ at $(q_0, 0)$ has the form $y_1=\psi(y_2, \lambda)$, and on it the differential equation takes the form

$$\dot{y}_2=J(y_2, \lambda).$$

In fact, in (1.8), the coordinate y_2 may be taken to agree on the center manifold with that defined here, and the functions J are then the same. Of course, $(\partial J/\partial y_2)(0,0)=0$. We now assume for simplicity that $(\partial J/\partial \lambda)(0,0) \neq 0$. Then the zero set of J near $(0,0)$ may be described by a function $\lambda=\lambda(y_2)$. Using this function we may parameterize the equilibria of $\dot{z}=h(z, \lambda)$ near $(q_0, 0)$ by y_2 as $(q(y_2), \lambda(y_2))$, with $q(0)=q_0$ and $\lambda(0)=0$. The equilibrium $(q(y_2), \lambda(y_2))$ has a unique invariant manifold $W(y_2)$ near Γ, tangent to the eigendirection for the most negative eigenvalue of its linearization. $W(y_2)$ is C^p, not C^∞. Also, near p_0 there is a hyperbolic saddle $p(\lambda)$ of $\dot{z}=h(z, \lambda)$ with unstable manifold $W^u(p(\lambda))$. The curves $W^u(p(\lambda(y_2)))$ and $W(y_2)$ meet Σ in points $z^1(y_2)$ and $z^2(y_2)$, respectively. Suppose Γ and Σ meet at $\tilde{z}$. We define the *separation function* $S(y_2)$ to be a convenient multiple of the signed distance between $z^1(y_2)$ and $z^2(y_2)$:

$$S(y_2)=\det(h(\tilde{z}, 0), z^1(y_2)-z^2(y_2)).$$

If we now set $\tilde{g}_1(y_2, \lambda)=S(y_2)$, $g_2(y_2, \lambda)=J(y_2, \lambda)$, then an equilibrium of $\dot{z}=h(z, \lambda)$ near q_0 corresponds to a zero of g_2, and a heteroclinic orbit from $p(\lambda)$ to a hyperbolic saddle near q_0 corresponds to a solution of $\tilde{g}_1=g_2=0$ at which $\partial g_2/\partial y_2<0$. We emphasize that in many problems, the function J is computable to arbitrary order, and the first derivatives (at least) of S are computable as Melnikov integrals with boundary terms; see [5]. In contrast, we do not know how to compute the coordinate changes that go into Vegter's approach.

An appropriate equivalence relation here is the following. Let $\mathscr{U}$ be the group of upper triangular 2×2 matrices with positive diagonal entries. We shall say that (g_1, g_2) is $\mathscr{U}$-equivalent to (f_1, f_2) provided

$$\begin{pmatrix} g_1(y_2, \lambda) \\ g_2(y_2, \lambda) \end{pmatrix} = \begin{pmatrix} A(y_2, \lambda) & B(y_2, \lambda) \\ 0 & C(y_2, \lambda) \end{pmatrix} \begin{pmatrix} f_1(Y(y_2, \lambda), \Lambda(\lambda)) \\ f_2(Y(y_2, \lambda), \Lambda(\lambda)) \end{pmatrix}, \tag{1.10}$$

where conditions (1.9) hold and A, B, C, Y, Λ are C^s for some s, $1 \leqslant s < p$. This equivalence relation takes the zero set of g_2 to that of f_2, and solutions of $g_1=g_2=0$ to those of $f_1=f_2=0$.

We remark that the function $\tilde{g}_1$ defined above should be regarded as defined on $g_2^{-1}(0)$, and that the ring of functions on $g_2^{-1}(0)$ is the ring of functions on $\mathbb{R}^2$ modulo the ideal generted by g_2. This remark provides an alternate explanation for the presence of B in (1.10). It also suggests the idea for how to define $\tilde{g}_1$ when $(\partial J/\partial \lambda)(0, 0) = 0$. Simply extend $\dot{z} = h(z, \lambda)$ to $\dot{z} = H(z, \lambda, \alpha)$, $\alpha \in \mathbb{R}$, $H(z, \lambda, 0) = h(z, \lambda)$, in such a way that $(\partial J/\partial \alpha)(0, 0, 0) \neq 0$. The equilibria near $(q_0, 0, 0)$ are then parameterized by (y_2, λ) as $(q(y_2, \lambda), \lambda, \alpha(y_2, \lambda))$, so we may define a separation function $S(y_2, \lambda)$. We then set $\tilde{g}_1 = S$.

In [5], where we considered heteroclinic bifurcation problems without a distinguished parameter, we defined $\tilde{g}_1$ and g_2 using Liapunov–Schmidt reduction of $g = 0$ instead of center manifold reduction. This and the simplicity of the situation enabled us to avoid C^p singularity theory. With a distinguished parameter C^p singularity theory apparently cannot be avoided, so there is not much advantage to Liapunov–Schmidt reduction.

Motivated by $\mathscr{D}$- and $\mathscr{U}$-equivalence, the singularity theory portion of this paper will be presented in the following generality. We consider C^p functions $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ and a Lie subgroup $\mathscr{G}$ of the $m \times m$ matrices such that $\mathscr{G}$ is an open subset of some vector subspace of the $m \times m$ matrices. We use x (instead of y_2) to denote the first argument of f. We say g is $(\mathscr{G}, s)$-equivalent to f if there exist C^s functions $S(x, \lambda)$ into $\mathscr{G}$, $X(x, \lambda)$ into $\mathbb{R}^n$, $\Lambda(\lambda)$ into $\mathbb{R}$, with $X(0, 0) = 0$, $\det D_x X(0, 0) > 0$, $\Lambda(0) = 0$, $\Lambda'(0) > 0$, such that

$$g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda)).$$

Note that if $\mathscr{G}$ is taken to be the group of $m \times m$ matrices with positive determinant and $p = s = \infty$, we obtain Golubitsky–Schaeffer bifurcation theory. Thus the present paper explains how to use Golubitsky–Schaeffer theory when functions are only C^p.

We now preview the rest of this paper.

Sections 2, 3, and 4 deal with C^p singularity theory. In Section 2 we define some of the algebraic objects that arise, and we prove some preliminary lemmas. In Section 3 we treat the C^p recognition problem, i.e., the problem of recognizing when a given function is equivalent to a given normal form. Our treatment is inspired by that of the C^∞ recognition problem in [3]. In Section 4 we study C^p universal unfoldings. Here the work of Vegter [8] carries over easily to the distinguished parameter context; we give some details for the sake of completeness. The most important results of Sections 3 and 4 (Corollaries 3.11 and 4.4) say that for C^∞ normal forms, C^∞ recognition criteria and universal unfoldings are in fact valid for C^p functions under C^s equivalence, provided s and $p - s$ are sufficiently large.

Section 5 applies C^p singularity theory to heteroclinic bifurcation problems. We give normal forms, recognition criteria, and universal unfoldings for some low codimension bifurcations under $\mathscr{D}$- and $\mathscr{U}$-equivalence, and we draw the corresponding heteroclinic bifurcation diagrams. In a companion paper [6] we treat a somewhat more complicated normal form than those treated here, related to pitchfork bifurcation with a heteroclinic orbit from a distant hyperbolic saddle.

2. Preliminaries

If $\mathscr{R}$ is a ring and M is a set, $\mathscr{R} \cdot M$ denotes the set of all finite linear combinations $\sum_{i=1}^{a} g_i f_i$ with $g_i \in \mathscr{R}$ and $f_i \in M$. If M is the finite set $\{f_1, ..., f_a\}$, we also denote $\mathscr{R} \cdot M$ by $\mathscr{R}\{f_1, ..., f_a\}$. $(\mathscr{R})^m$ denotes the product $\mathscr{R} \times \cdots \times \mathscr{R}$ (m times).

From now on, $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, and α is a vector of parameters. We let $\mathscr{E}^p_{x,\lambda}$, $1 \leqslant p \leqslant \infty$, denote the ring of germs at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ of C^p-functions $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Then $(\mathscr{E}^p_{x,\lambda})^m$ can be identified with the set of germs at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ of C^p functions $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, a module over $\mathscr{E}^p_{x,\lambda}$. Note that $(\mathscr{E}^p_{x,\lambda})^m = \mathscr{E}^p_{x,\lambda}\{e_1, ..., e_m\}$, where $e_i\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is the constant function whose value is the ith standard basis vector of $\mathbb{R}^m$.

We let $\mathscr{M}^p_{x,\lambda} = \mathscr{E}^p_{x,\lambda}\{x_1, ..., x_n, \lambda\}$, an ideal in $\mathscr{E}^p_{x,\lambda}$ (*not* the maximal ideal unless $p = \infty$). For each $(n+1)$-tuple $(\omega_1, ..., \omega_{n+1})$ of nonnegative integer, let

$$|\omega| = \omega_1 + \cdots + \omega_{n+1},$$
$$(x, \lambda)^\omega = x_1^{\omega_1} \cdots x_n^{\omega_n} \lambda^{\omega_{n+1}}.$$

Then we define

$$(\mathscr{M}^p_{x,\lambda})^{(k)} = \mathscr{M}^p_{x,\lambda} \cdot \cdots \cdot \mathscr{M}^p_{x,\lambda} \ (k \text{ times}) = \mathscr{E}^p_{x,\lambda}\{(x, \lambda)^\omega : |\omega| = k\}.$$

If $\alpha \in \mathbb{R}^k$, we shall also have cause to consider $\mathscr{E}^p_{x,\alpha}$, the ring of germs at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^k$ of C^p maps $f\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$; $\mathscr{E}^p_{x,\lambda,\alpha}$; etc.

Let $M(m)$ be the space of all $m \times m$ matrices, let $\mathscr{S}$ be a vector subspace of $M(m)$, and let $\mathscr{G}$ be a Lie group that is an open subset of $\mathscr{S}$. Then for $1 \leqslant p \leqslant \infty$ we define $\mathscr{S}^p_{x,\lambda}$ (resp. $\mathscr{G}^p_{x,\lambda}$) to be the set of germs at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ of C^p maps $S\colon \mathbb{R}^n \times \mathbb{R} \to \mathscr{S}$ (resp. $\mathscr{G}$).

Lemma 2.1. *$\mathscr{S}$ is closed under multiplication.*

Proof. Since the multiplication map $M(m) \times M(m) \to M(m)$ is analytic and takes $\mathscr{G} \times \mathscr{G}$ into $\mathscr{S}$, and $\mathscr{G}$ is open in $\mathscr{S}$, it follows that multiplication takes $\mathscr{S} \times \mathscr{S}$ into $\mathscr{S}$. ∎

Lemma 2.1 implies that $\mathscr{S}^p_{x,\lambda}$ may be regarded as a ring. Thus if $f \in (\mathscr{E}^p_{x,\lambda})^m$, then

$$\mathscr{S}^p_{x,\lambda}\{f\} = \{S \cdot f : S \in \mathscr{S}^p_{x,\lambda}\}.$$

Let $S_1, ..., S_a$ be a basis for $\mathscr{S}$, and also let S_i denote the constant function from $\mathbb{R}^n \times \mathbb{R}$ to $\mathscr{S}$ whose value is S_i. The $\mathscr{S}^p_{x,\lambda}\{f\} = \mathscr{E}^p_{x,\lambda}\{S_1 f, ..., S_a f\}$.

LEMMA 2.2. *$\mathscr{G}$ is semialgebraic (i.e., described by a finite number of polynomial equalities and inequalities).*

Proof. By Lemma 2.1, $Gl(m) \cap \mathscr{S}$ is a Lie group, and it is obviously a semialgebraic set. Since $Gl(m) \cap \mathscr{S}$ is semialgebraic, it has a finite number of components, each of which is semialgebraic [1, p. 31]. Since $\mathscr{G}$ is an open subgroup of $Gl(m) \cap \mathscr{S}$, $\mathscr{G}$ is closed in $Gl(m) \cap \mathscr{S}$. Therefore $\mathscr{G}$ is a finite union of the components of $Gl(m) \cap \mathscr{S}$, so $\mathscr{G}$ is semialgebraic. ∎

Consider the initial value problem

$$\begin{aligned} &\dot{X}(t) = X(t)\, S(t), \qquad t \in I \\ &X(0) \in \mathscr{G}, \end{aligned} \tag{2.1}$$

where $S(t) \in \mathscr{S}$ for all t in an interval I containing 0, and $S(t)$ is continuous.

LEMMA 2.3. *If $X(t)$ is the solution of* (2.1), *then $X(t) \in \mathscr{G}$ for all $t \in I$.*

Proof. If $X(t) \in \mathscr{S}$ then $X(t)\, S(t) \in \mathscr{S}$ by Lemma 2.1, so $\mathscr{S}$ is invariant under the flow of (2.1). Since (2.1) is a linear differential equation, $X(t)$ is defined for all $t \in I$ and is invertible for all $t \in I$. Thus $X(t)$ belongs to the Lie group $Gl(m) \cap \mathscr{S}$ for all $t \in I$. But $\mathscr{G}$ is a union of components of $Gl(m) \cap \mathscr{S}$ as in the proof of Lemma 2.2. Therefore $X(t) \in \mathscr{G}$ for all $t \in I$. ∎

3. RECOGNITION

Let $f, g \in (\mathscr{E}^p_{x,\lambda})^m$. Let $\mathscr{S}$ be a subspace of $M(m)$, and let $\mathscr{G}$ be a Lie group that is an open subset of $\mathscr{S}$. We say that g is *$(\mathscr{G}, s)$-equivalent to f*, and write $g \sim^s_{\mathscr{G}} f$, provided

$$g(x, \lambda) = S(x, \lambda)\, f(X(x, \lambda), \Lambda(\lambda)), \tag{3.1}$$

where $S \in \mathscr{G}^s_{x,\lambda}$, $X \in (\mathscr{E}^s_{x,\lambda})^n$, $\Lambda \in \mathscr{E}^s_\lambda$, $X(0, 0) = 0$, $\det D_x X(0, 0) > 0$, $\Lambda(0) = 0$, $\Lambda'(0) > 0$. Alternatively, we let $\gamma = (S, X, \Lambda)$ and write $g = \gamma(f)$. We say γ is a *polynomial* if S, X, and Λ are polynomials.

Denote the inverse of

$$(x, \lambda) \mapsto (X(x, \lambda), \Lambda(\lambda)) = (y, \mu)$$

by

$$(y, \mu) \mapsto (X^{-1}(y, \mu), \Lambda^{-1}(\mu)) = (x, \lambda).$$

Of course X^{-1} and Λ^{-1} are C^s by the inverse function theorem. Then $g = \gamma(f)$ if and only if

$$f(y, \mu) = S^{-1}(X^{-1}(y, \mu), \Lambda^{-1}(\mu))\, g(X^{-1}(y, \mu), \Lambda^{-1}(\mu)),$$

where S^{-1} is just the inverse of the matrix S. Thus we define

$$\gamma^{-1} = (S^{-1}(X^{-1}(y, \mu), \Lambda^{-1}(\mu)), X^{-1}, \Lambda^{-1}),$$

and note that $g = \gamma(f)$ if and only if $f = \gamma^{-1}(g)$. It follows easily that $\sim^s_{\mathscr{G}}$ is an equivalence relation.

Suppose $f \in (\mathscr{E}^{\infty}_{x,\lambda})^m$ is a "normal form"; in practice, f is always a polynomial. To decide which $g \in (\mathscr{E}^{\infty}_{x,\lambda})^m$ are $(\mathscr{G}, \infty)$-equivalent to f, one would like to write Eq. (3.1), expand everything as Taylor series about $(0, 0)$, and decide for which g one can solve for the coefficients in the series for S, X, and Λ. In order to make the calculation finite and avoid considering the C^{∞}-flat "tail," one would like to identify a finite codimension $\mathscr{E}^{\infty}_{x,\lambda}$-submodule M^{∞} of $(\mathscr{E}^{\infty}_{x,\lambda})^m$ such that the calculation can be done modulo M^{∞}. The result of such a calculation would be a finite number of conditions on the first few derivatives of g at $(0, 0)$ such that $g \sim^s_{\mathscr{G}} f$ if and only if the conditions are satisfied. This is the strategy followed in [3, 4, 2]. Our goal is to show that if s and $p - s$ are sufficiently large, then $g \in (\mathscr{E}^p_{x,\lambda})^m$ is $(\mathscr{G}, s)$-equivalent to f if and only if exactly the same conditions on the first few derivatives of g are satisfied. This is the content of Corollary 3.11 below, toward which the present section aims.

Let $f \in (\mathscr{E}^p_{x,\lambda})^m$. For $1 \leqslant s \leqslant \infty$, define the *$(\mathscr{G}, s)$-restricted tangent space* of f to be

$$RT^s_{\mathscr{G}}(f) = \mathscr{S}^s_{x,\lambda}\{f\} + \mathscr{M}^s_{x,\lambda}\left\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\},$$

and $\mathscr{E}^s_{x,\lambda}$-module. Note that if we define a one-parameter perturbation of f by

$$f(x, \lambda, t) = S(x, \lambda, t)\, f(X(x, \lambda, t), \lambda)$$

with

$$S \in \mathcal{G}^{\infty}_{x,\lambda,t},\ X \in (\mathcal{E}^{\infty}_{x,\lambda,t})^n,$$
$$S(x, \lambda, 0) = I,\ X(x, \lambda, 0) = x, \text{ and } X(0, 0, t) = 0,$$

then $(\partial f/\partial t)(x, \lambda, 0)$ is in $RT^{\infty}_{\mathcal{G}}(f)$.

We may choose generators for $RT^s_{\mathcal{G}}(f)$ as follows. Let $S_1, ..., S_a$ be a basis for $\mathcal{S}$ with $S_1 = I$. Then $RT^s_{\mathcal{G}}(f)$ is generated by

$$\{S_1 f, ..., S_a f\} \cup \left\{x_i \frac{\partial f}{\partial x_j} : 1 \leqslant i, j \leqslant n\right\} \cup \left\{\lambda \frac{\partial f}{\partial x_j} : 1 \leqslant j \leqslant n\right\}.$$

Denote this set of generators by $z_1(f), ..., z_N(f)$, and note that $z_1(f) = f$.

LEMMA 3.1. *If $f, h \in (\mathcal{E}^p_{x,\lambda})^m$ and $RT^s_{\mathcal{G}}(f + th) = RT^s_{\mathcal{G}}(f)$ for all $t \in [0, 1]$, then $f + h \sim^s_{\mathcal{G}} f$.*

Remark. The equivalence has $\Lambda =$ identity, but this is not important in the sequel.

Proof. Let $F(x, \lambda, t) = f(x, \lambda) + th(x, \lambda)$. For $t_1, t_2 \in [0, 1]$, define $t_1 \sim t_2$ if $f + t_1 h \sim^s_{\mathcal{G}} f + t_2 h$. We claim that for each t_1, $t_1 \sim t_2$ for all t_2 in a neighborhood of t_1 in $[0, 1]$. Since $[0, 1]$ is connected, it follows that $0 \sim 1$, which proves the lemma. We shall prove the claim for $t_1 = 0$; the proof for other t_1 is the same.

We want to find mappings $S(x, \lambda, t)$ and $X(x, \lambda, t)$, $t \in [0, \varepsilon)$, such that

$$F(x, \lambda, 0) = S(x, \lambda, t)\, F(X(x, \lambda, t), \lambda, t) \tag{3.2}$$

with

$$\begin{gathered} S \in \mathcal{G}^s_{x,\lambda,t},\ X \in (\mathcal{E}^s_{x,\lambda,t})^n, \\ S(x, \lambda, 0) = I,\ X(x, \lambda, 0) = x, \text{ and } X(0, 0, t) = 0. \end{gathered} \tag{3.3}$$

Let a dot denote partial derivative with respect to t. Then equivalently, S and X must satisfy (3.3) and

$$0 = \dot{S}F(X, \lambda, t) + S\{D_x F(X, \lambda, t)\, \dot{X} + h(X, \lambda)\}. \tag{3.4}$$

Suppose there exist $u \in \mathcal{S}^s_{x,\lambda,t}$ and $v \in (\mathcal{E}^s_{x,\lambda,t})^n$ with $v(0, 0, t) = 0$ such that

$$-h(x, \lambda) = u(x, \lambda, t)\, F(x, \lambda, t) + D_x F(x, \lambda, t)\, v(x, \lambda, t). \tag{3.5}$$

If we successively define X and S by

$$\dot{X} = v(X(x, \lambda, t), \lambda, t), \qquad X(x, \lambda, 0) = x,$$
$$\dot{S} = S(x, \lambda, t)\, u(X(x, \lambda, t), \lambda, t), \qquad S(x, \lambda, 0) = I,$$

then (3.3) and (3.4) are satisfied. (The fact that $S(x, \lambda, t) \in \mathscr{G}$ follows from Lemma 2.3).

To see that u and v satisfying (3.5) exist, note that for $i = 1, ..., N$, $z_i(f + th) = z_i(f) + tz_i(h)$. Let $t_0 \in (0, 1]$. By assumption, for $i = 1, ..., N$ we have

$$z_i(f) + t_0 z_i(h) = \sum_{j=1}^{N} w_{ij} z_j(f), \qquad w_{ij} \in \mathscr{E}^s_{x,\lambda}. \tag{3.6}$$

Letting $z(f) = (z_1(f), ..., z_N(f))^{\mathrm{T}}$, we have from (3.6)

$$z(h) = Qz(f),$$

where Q is a $n \times n$ matrix with entries in $\mathscr{E}^s_{x,\lambda}$. Therefore,

$$z(h) = Q(z(F) - tz(h)),$$

so

$$z(h) = (I + tQ)^{-1} Qz(F).$$

(Notice $I + tQ$ is invertible for t small.) The first equation of this system is

$$h = \sum_{j=1}^{N} \tilde{w}_{1j}\, z_j(f), \qquad \tilde{w}_{1j} \in \mathscr{E}^s_{x,\lambda,t},$$

which implies that (3.5) has a solution (u, v) of the desired form. ∎

Let M be an $\mathscr{E}^s_{x,\lambda}$-submodule of $(\mathscr{E}^s_{x,\lambda})^m$, and let $f \in \mathscr{E}^p_{x,\lambda}$. We say that M is *$(r, s, p, \mathscr{G})$-ignorable for f* provided: if $g \in (\mathscr{E}^p_{x,\lambda})^m$, then $g \sim^{s-1}_{\mathscr{G}} f$ if and only if there is a polynomial γ of degree $\leqslant r$ such that $\gamma(f) = g - h$ with $h \in M$. (The reason for using $s - 1$ rather than s in this definition does not appear until the end of the proof of Theorem 3.10 below.) We say that M is *$(\mathscr{G}, s)$-intrinsic* if $h_1 \in M$ and $h_2 \sim^{s+1}_{\mathscr{G}} h_1$ imply $h_2 \in M$. (The reason for using $s + 1$ rather than s here appears in the proof of Proposition 3.4.)

THEOREM 3.2. *Let $f \in (\mathscr{E}^p_{x,\lambda})^m$ and let M be an $(\mathscr{E}^s_{x,\lambda})$-submodule of $(\mathscr{E}^s_{x,\lambda})^m$. Assume:*

(1) *M is $(\mathscr{G}, s)$-intrinsic;*

(2) *M is generated by a finite number of homogeneous polynomial maps* $u_1, \ldots, u_a$;

(3) *for all* $h \in M$, $RT_{\mathscr{G}}^{s-1}(f+h) = RT_{\mathscr{G}}^{s-1}(f)$;

(4) $(\mathscr{E}^p_{x,\lambda})^m \subset M + \mathbb{R}\{v_1, \ldots, v_b\}$, *where* $v_1, \ldots, v_b$ *are polynomial maps of degree* $\leqslant r$;

(5) $s-1 \geqslant r$.

Then M is $(r, s, p, \mathscr{G})$-ignorable for f.

Proof. First assume that there is a polynomial γ such that $\gamma(f) = g - h$, where $g \in (\mathscr{E}^p_{x,\lambda})^m$ and $h \in M$. Then γ^{-1} is analytic, and

$$\gamma^{-1}(g) = \gamma^{-1}(g-h) + \gamma^{-1}(h) = f + \tilde{h}, \qquad \tilde{h} \in M.$$

Since $\tilde{h} \in M$, $t\tilde{h} \in M$ for all t. Thus $g \sim_{\mathscr{G}}^{\infty} f + \tilde{h}$, and by (3) and Lemma 3.1, $f + \tilde{h} \sim_{\mathscr{G}}^{s-1} f$. Therefore $g \sim_{\mathscr{G}}^{s-1} f$.

Now assume that $g \in (\mathscr{E}^p_{x,\lambda})^m$ and $g = \tilde{\gamma}(f)$, where $\tilde{\gamma}$ is C^{s-1}. Let γ be the r-jet (at the origin) of $\tilde{\gamma}$, a polynomial. Let $h = g - \gamma(f)$. It is not hard to see that the r-jet of h is zero. Now h is in $(\mathscr{E}^p_{x,\lambda})^m$; therefore (2) and (4) imply that

$$h = \sum_{i=1}^{a} w_i u_i + \sum_{j=1}^{b} c_j v_j,$$

where $w_i \in \mathscr{E}^s_{x,\lambda}$ and $c_j \in \mathbb{R}$. If $\deg(u_i) \leqslant r$, let $\tilde{w}_i$ equal w_i minus its jet of order $r - \deg(u_i)$; otherwise let $\tilde{w}_i = w_i$. Then since the r-jet of h is zero,

$$h = \sum_{i=1}^{a} \tilde{w}_i u_i.$$

Therefore $h \in M$. Since $\gamma(f) = g - h$, the proof is complete. ∎

Theorem 3.2 motivates an attempt to identify $(\mathscr{G}, s)$-intrinsic submodules of $(\mathscr{E}^s_{x,\lambda})^m$. We do this with the aid of s-intrinsic ideals.

Let $\mathscr{I}$ be an ideal in $\mathscr{E}^s_{x,\lambda}$. $\mathscr{I}$ is *s-intrinsic* if $h \in \mathscr{I}$ implies $h(X(x, \lambda), \Lambda(\lambda)) \in \mathscr{I}$ whenever

$$X \in (\mathscr{E}^{s+1}_{x,\lambda})^n, \ \Lambda \in \mathscr{E}^{s+1}_{\lambda}, \ X(0,0) = 0, \ \Lambda(0) = 0.$$

(The reason for using $s+1$ in this definition appears in the proof of Proposition 3.4 below.)

PROPOSITION 3.3. *Sums and products of s-intrinsic ideals are s-intrinsic.*

The proof is left to the reader.

PROPOSITION 3.4. *$\mathcal{M}^s_{x,\lambda}$ and $\mathcal{E}^s_{x,\lambda}\{\lambda\}$ are s-intrinsic ideals.*

Proof. We give the proof only for $\mathcal{M}^s_{x,\lambda}$. Let $h \in \mathcal{M}^s_{x,\lambda}$ and let $X(x, \lambda)$ and $\Lambda(\lambda)$ be C^{s+1} with $X(0, 0) = 0$ and $\Lambda(0) = 0$. Write $X(x, \lambda) = (X_1(x, \lambda), ..., X_n(x, \lambda))$. Then

$$h(x, \lambda) = \sum_{i=1}^{n} x_i h_i(x, \lambda) + \lambda h_{n+1}(x, \lambda), \qquad h_i \in \mathcal{E}^s_{x,\lambda}.$$

Therefore

$$\begin{aligned} h(X(x, \lambda), \Lambda(\lambda)) = \sum_{i=1}^{n} X_i(x, \lambda)\, h_i(X(x, \lambda), \Lambda(\lambda)) \\ + \Lambda(\lambda)\, h_{n+1}(X((x, \lambda), \Lambda(\lambda)). \end{aligned} \tag{3.7}$$

But

$$\begin{aligned} X_i(x, \lambda) &= \sum_{i=1}^{n} x_j u_{ij}(x, \lambda) + \lambda u_{i,n+1}(x, \lambda), \qquad && u_{ij} \in \mathcal{E}^s_{x,\lambda}; \\ \Lambda(\lambda) &= \lambda v(\lambda), && v \in \mathcal{E}^s_{\lambda}. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) shows that $h(X(x, \lambda), \Lambda(\lambda)) \in \mathcal{M}^s_{x,\lambda}$. ∎

COROLLARY 3.5. *Ideals of the form*

$$\mathcal{I}^s = (\mathcal{M}^s_{x,\lambda})^{(k)} + (\mathcal{M}^s_{x,\lambda})^{(k_1)}\{\lambda^{l_1}\} + \cdots + (\mathcal{M}^s_{x,\lambda})^{(k_t)}\{\lambda^{l_t}\}, \tag{3.9}$$

with $0 < l_1 < \cdots < l_t$ and $k > k_1 + l_1 > \cdots > k_t + l_t$, are s-intrinsic.

Note that ideals of the form (3.9) are generated over $\mathcal{E}^s_{x,\lambda}$ by homogeneous monomials.

Let $\mathcal{I}^s_1, ..., \mathcal{I}^s_c$ be s-intrinsic ideals. For each $i = 1, ..., c$, let J_i be a finite set and let $\{v_{ij}: j \in J_i\}$ be a collection of constant functions from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^m$. Let

$$M^s = \sum_{i=1}^{c} \mathcal{I}^s_i\{v_{ij}: j \in J_i\}. \tag{3.10}$$

A $(\mathcal{G}, s)$-intrinsic submodule of $(\mathcal{E}^s_{x,\lambda})^m$ is called *simple* if it can be written in the form (3.10) with each $\mathcal{I}^s_i$ an s-intrinsic ideal of the form (3.9).

EXAMPLE 3.6. For any $\mathcal{G}$, if $\mathcal{I}^s$ has the form (3.9), then $(\mathcal{I}^s)^m$ is a simple $(\mathcal{G}, s)$-intrinsic submodule of $(\mathcal{E}^s_{x,\lambda})^m$.

Let $m = 2$. Recall from Section 1 that $\mathcal{D}$ is the group of 2×2 diagonal matrices with positive diagonal terms, and $\mathcal{U}$ is the group of 2×2 upper triangular matrices with positive diagonal terms. Let $\{e_1, e_2\}$ denote the standard basis of $\mathbb{R}^2$ and also the respective constant functions of (x, λ). Let $\mathscr{I}_1^s$ and $\mathscr{I}_2^s$ have the form (3.9).

EXAMPLE 3.7. $\mathscr{I}_1^s\{e_1\} \oplus \mathscr{I}_2^s\{e_2\}$ is a simple $(\mathscr{D}, s)$-intrinsic submodule of $(\mathscr{E}_{x,\lambda}^s)^2$.

EXAMPLE 3.8. $\mathscr{I}_1^s\{e_1\} + \mathscr{I}_2^s\{e_1, e_2\}$ is a simple $(\mathscr{U}, s)$-intrinsic submodule of $(\mathscr{E}_{x,\lambda}^s)^2$.

The easy verifications of these examples are left to the reader; they use only the fact that the ideals are s-intrinsic. Example 3.7 may be generalized as follows. (The generalization will not be used in the sequel.) Generalizations of Example 3.8 are left to the reader.

PROPOSITION 3.9. *Suppose* $\mathbb{R}^m = V_1 \oplus \cdots \oplus V_c$ *where each* V_i *is* $\mathscr{G}$*-invariant. Let a basis for* V_i *be* $\{v_{ij}: j \in J_i\}$, *and let* v_{ij} *also denote the corresponding constant function of* (x, λ). *Let* $\mathscr{I}_1^s, \ldots, \mathscr{I}_c^s$ *have the form* (3.9), *and let*

$$M = \bigoplus_{i=1}^{c} \mathscr{I}_i^s\{v_{ij}: j \in J_i\}.$$

Then M *is a simple* $(\mathscr{G}, s)$*-intrinsic submodule of* $(\mathscr{E}_{x,\lambda}^s)^m$.

Given ideals $\mathscr{I}_1^\infty, \ldots, \mathscr{I}_c^\infty$ of the form (3.9) and a simple $(\mathscr{G}, \infty)$-intrinsic submodule of $(\mathscr{E}_{x,\lambda}^\infty)^m$, namely

$$M^\infty = \sum_{i=1}^{c} \mathscr{I}_i^\infty\{v_{ij}: j \in J_i\},$$

we obtain corresponding simple $(\mathscr{G}, s)$-intrinsic submodules of $(\mathscr{E}_{x,\lambda}^s)^m$, namely

$$M^s = \sum_{i=1}^{c} \mathscr{I}_i^s\{v_{ij}: j \in J_i\}.$$

This idea is used in the following result.

THEOREM 3.10. *Let* $f \in (\mathscr{E}_{x,\lambda}^\infty)^m$. *Let* M^∞ *be a simple* $(\mathscr{G}, \infty)$*-intrinsic submodule of* $(\mathscr{E}_{x,\lambda}^\infty)^m$. *Assume*

$$\text{(i)} \quad ((\mathscr{M}_{x,\lambda}^\infty)^{(r+1)})^m \subset M^\infty,$$

and

$$\text{(ii)} \quad M_\infty \subset \mathscr{M}_{x,\lambda}^\infty \cdot RT_{\mathscr{G}}^\infty(f).$$

If $s \geqslant r+1$ *and* $p \geqslant r+s+1$, *then* M^s, *the corresponding simple* $(\mathscr{G}, s)$-*intrinsic submodule of* $(\mathscr{E}^s_{x,\lambda})^m$, *is* $(r, s, p, \mathscr{G})$-*ignorable for* f.

Proof. We verify the hypotheses (2) to (4) of Theorem 3.2 for M^s.

(2) M^s is generated by (generators of $\mathscr{I}^s_i$) $\cdot$ (constant maps), which are homogeneous polynomial maps by the comment following Corollary 3.5.

(4) Assumption (i) implies that M^∞ contains each generator of $((\mathscr{M}^\infty_{x,\lambda})^{(r+1)})^m$, and therefore that M^s contains $((\mathscr{M}^s_{x,\lambda})^{(r+1)})^m$. But for $g \in (\mathscr{E}^p_{x,\lambda})^m$, $p \geqslant r+1$, Taylor's theorem shows that

$$g = j^r(g) + \text{an element of } ((\mathscr{M}^{p-r-1}_{x,\lambda})^{(r+1)})^m,$$

where $j^r(g)$ denotes the r-jet of g at the origin. Thus, since $p-r-1 \geqslant s$,

$$g = j^r(g) + \text{an element of } ((\mathscr{M}^s_{x,\lambda})^{(r+1)})^m.$$

This implies (4), with $v_1, \ldots, v_b$ a set of generators for the polynomial maps of degree $\leqslant r$.

(3) Assumption (ii) implies that $M^s \subset \mathscr{M}^s_{x,\lambda} \cdot RT^s_{\mathscr{G}}(f)$. Let $h \in M^s$. Recall the generators of $RT^s_{\mathscr{G}}(h)$, $z_1(h), \ldots, z_N(h)$, defined earlier. By writing h in terms of the generators of M^s, which are homogeneous polynomial maps, one sees that for $i = 1, \ldots, N$, $z_i(h) \in M^{s-1}$. But $M^{s-1} \subset \mathscr{M}^{s-1}_{x,\lambda} \cdot RT^{s-1}_{\mathscr{G}}(f)$, so

$$z_i(h) = \sum_{j=1}^{N} w_{ij} z_j(f), \qquad w_{ij} \in \mathscr{M}^{s-1}_{x,\lambda}.$$

Therefore

$$z(f+h) = z(f) + z(h) = (I+B)\, z(f)$$

where B has entries in $\mathscr{M}^{s-1}_{x,\lambda}$. Therefore $RT^{s-1}_{\mathscr{G}}(f+h) \subset RT^{s-1}_{\mathscr{G}}(f)$. Inverting $I+B$ shows $RT^{s-1}_{\mathscr{G}}(f) \subset RT^{s-1}_{\mathscr{G}}(f+h)$. ∎

COROLLARY 3.11. *In the situation of Theorem* 3.10, *for* $g \in (\mathscr{E}^\infty_{x,\lambda})^m$, *there exist a finite number of polynomial equalities and inequalities involving the partial derivatives of* g *at* $(0,0)$ *of order* $\leqslant r$ *such that* $g \sim^\infty_{\mathscr{G}} f$ *if and only if these conditions are satisfied. Moreover, if* p *and* s *are as in Theorem* 3.10, *then* $g \in (\mathscr{E}^p_{x,\lambda})^m$ *is* $(\mathscr{G}, s)$-*equivalent to* f *if and only if exactly the same conditions are satisfied.*

Proof. If $g \in (\mathscr{E}^\infty_{x,\lambda})^m$, then by Theorem 3.10, $g \sim^\infty_{\mathscr{G}} f$ if and only if there is a polynomial γ of degree $\leqslant r$ such that

$$\gamma(f) = g - h \qquad \text{with} \quad h \in M^\infty. \tag{3.11}$$

Because of (1), (3.11) becomes

$$j^r\gamma(f)=j^r(g) \text{ modulo } M^\infty. \tag{3.12}$$

If $g\in(\mathscr{E}^p_{x,\lambda})^m$, then by Theorem 3.10, $g \sim^{s-1}_{\mathscr{G}} f$ if and only if there is a polynomial γ of degree $\leqslant r$ such that

$$\gamma(f)=g-h \qquad \text{with} \quad h\in M^s. \tag{3.13}$$

Since $g=j^r(g)+$ an element of M^s (see the verification of (4) in the proof of Theorem 3.10) and $\gamma(f)=j^r\gamma(f)+$ an element of M^∞, (3.13) becomes

$$j^r\gamma(f)=j^r(g) \text{ modulo } M^s. \tag{3.14}$$

Since $s\geqslant r+1$, (3.12) and (3.14) give the same equations for $j^r(g)$. With the help of Lemma 2.2, we see that the set of pairs $(\gamma, j^r(g))$ for which (3.12) or (3.14) is true, and for which the polynomial $\gamma=(S, X, \Lambda)$ satisfies the conditions for a $(\mathscr{G}, \infty)$-equivalence, is semialgebraic. By the Tarski–Seidenberg Theorem, the projection of this set to $j^r(g)$-space is semialgebraic. ∎

We remark that Refs. [3] and [2], working with $(Gl(m), \infty)$-equivalence on $(\mathscr{E}^\infty_{x,\lambda})^m$, define submodules larger than M^∞ modulo which the C^∞ recognition problem can be solved. (Of course, the larger the submodule, the easier the calculation.) According to Corollary 3.11, the result of such a simplified calculation is valid for $(Gl(m), s)$-equivalence on $(\mathscr{E}^p_{x,\lambda})^m$ provided p and s satisfy by conditions of Theorem 3.10. Note, however, that the number r in these conditions is calculated for the submodule M^∞, and not for the larger submodule of [3] or [2].

4. Unfoldings

We shall call a function $\sigma: \mathbb{R}\to\mathbb{R}$ a *deficiency function* or *d-function* provided σ is non-decreasing, finite-to-one, and $\sigma(p)\leqslant p$ for all p. If $\sigma(p)\leqslant p-1$ for all p, and m and b are positive integers, one can define new d-functions $\tilde{\phi}^{(m,b;\sigma)}_n(p)$ and $\tilde{\psi}^{(m,b)}_n(p)$ (the latter does not depend on σ) such that the following theorem is true

THEOREM 4.1 C^p Malgrange Preparation Theorem. *Let σ be a d-function with $\sigma(p)\leqslant p-1$. Let $x\in\mathbb{R}^n$ and $a\in\mathbb{R}^k$. Let $f_1, ..., f_a$, $g_1, ..., g_b \in (\mathscr{E}^{p_0}_{x,\alpha})^m$. Let $p_1=\min\{p: \sigma(p)>0\}$, $p_2=\min\{p: \tilde{\phi}^{(m,b;\sigma)}_n(p)>0\}$, and assume $p_1\leqslant p_0$, $p_2\leqslant p_0$. Assume that for all p such that $p_1\leqslant p\leqslant p_0$,*

$$(\mathscr{E}^p_{x,\alpha})^m\subset\mathscr{E}^{\sigma(p)}_{x,\alpha}\{f_1, ..., f_a\}+(\mathscr{E}^{p-1}_{x,\alpha}\{\alpha_1, ..., \alpha_k\})^m+\mathbb{R}\{g_1, ..., g_b\}.$$

Then for all p such that $p_2 \leqslant p \leqslant p_0$,

$$(\mathscr{E}^p_{x,\alpha})^m \subset \mathscr{E}^{\tilde{\phi}_n^{(m,b;\sigma)}(p)}_{x,\alpha}\{f_1, ..., f_a\} + \mathscr{E}^{\tilde{\psi}_n^{(m,b)}(p)}_{x,\alpha}\{g_1, ..., g_b\}.$$

The definitions of $\tilde{\phi}_n$ and $\tilde{\psi}_n$, and the proof of Theorem 4.1, are deferred until the end of this section. They are based on [8].

Let $f \in (\mathscr{E}^{q+1}_{x,\lambda})^m$ and let $\alpha \in \mathbb{R}^k$. Then $F \in (\mathscr{E}^p_{x,\lambda,\alpha})^m$ is a C^p *unfolding of* f provided $F(x, \lambda, 0) = f(x, \lambda)$. Let $F(x, \lambda, \alpha)$ and $G(x, \lambda, \beta)$ be C^p unfoldings of f. Let $\mathscr{G}$ and $\mathscr{S}$ be as in Sections 2 and 3. We make the following definitions.

We say G $(\mathscr{G}, s)$*-factors through* F if there are C^s maps $S(x, \lambda, \beta)$ into $\mathscr{G}$, $X(x, \lambda, \beta)$ into $\mathbb{R}^n$, $\Lambda(\lambda, \beta)$ into $\mathbb{R}$, and $A(\beta)$ into α-space, with $S(x, \lambda, 0) = I$, $X(x, \lambda, 0) = x$, $\Lambda(\lambda, 0) = \lambda$, $A(0) = 0$, such that

$$G(x, \lambda, \beta) = S(x, \lambda, \beta)\, F(X(x, \lambda, \beta), \Lambda(\lambda, \beta), A(\beta)).$$

We say F is $(\mathscr{G}, s, p)$*-versal* if every C^p unfolding G of f $(\mathscr{G}, s)$-factors through F.

We let

$$\tilde{T}^s_{\mathscr{G}}(f) = \mathscr{S}^s_{x,\lambda}\{f\} + \mathscr{E}^s_{x,\lambda}\left\{\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right\},$$

an $\mathscr{E}^s_{x,\lambda}$-submodule of $(\mathscr{E}^s_{x,\lambda})^m$. We define the $(\mathscr{G}, s)$*-tangent space of* f to be

$$T^s_{\mathscr{G}}(f) = \tilde{T}^s_{\mathscr{G}}(f) + \mathscr{E}^s_\lambda\left\{\frac{\partial f}{\partial \lambda}\right\}.$$

Note that if we define a one-parameter perturbation of f by $f(x, \lambda, t) = S(x, \lambda, t)\, f(X(x, \lambda, t), \Lambda(\lambda, t))$ with $S \in \mathscr{G}^\infty_{x,\lambda,t}$, $X \in (\mathscr{E}^\infty_{x,\lambda,t})^n$, $\Lambda \in \mathscr{E}^\infty_{\lambda,t}$, $S(x, \lambda, 0) = I$, $X(x, \lambda, 0) = x$, $\Lambda(\lambda, 0) = \lambda$, then $(\partial f/\partial t)(x, \lambda, 0)$ is in $T^\infty_{\mathscr{G}}(f)$.

THEOREM 4.2. *Let* $f \in (\mathscr{E}^{q+1}_{x,\lambda})^m$. *Let* $\sigma: \mathbb{R} \to \mathbb{R}$ *be a d-function. Let* $p_1 = \min\{p: \sigma(p) > 0\}$, *and assume* $p_1 \leqslant q$. *Let* $F(x, \lambda, \alpha)$, $\alpha = (\alpha_1, ..., \alpha_k)$, *be a* C^{q+1} *unfolding of F. Assume there exist* $h_1, ..., h_c \in \mathscr{E}^q_\lambda\{\partial f/\partial \lambda\}$ *such that for* $p_1 \leqslant p \leqslant q$,

$$(\mathscr{E}^p_{x,\lambda})^m \subset \tilde{T}^{\sigma(p)}_{\mathscr{G}}(f) + \mathbb{R}\left\{h_1, ..., h_c, \frac{\partial F}{\partial \alpha_1}(x, \lambda, 0), ..., \frac{\partial F}{\partial \alpha_k}(x, \lambda, 0)\right\}.$$

Let $p_2 = \min\{p: \tilde{\phi}^{(m,c+k;\sigma)}_{n+1}(p) > 0\}$. *Then for all p such that* $p_2 \leqslant p \leqslant q$, *F is* $(\mathscr{G}, \tilde{\phi}^{(m,c+k;\sigma)}_{n+1}(p), p+1)$*-versal.*

EXAMPLE 4.3. We take $n=m=1$, $\mathscr{G}=(0,\infty)$, $f(x,\lambda)=x^3-\lambda x$. For any s,

$$\tilde{T}^s_{\mathscr{G}}(f)=\mathscr{E}^s_{x,\lambda}\{x^3-\lambda x, 3x^2-\lambda\}.$$

Then

$$x^3, \lambda x, \text{ and } \lambda^2 \text{ are in } \tilde{T}^s_{\mathscr{G}}(f) \text{ for any } s, \tag{4.1}$$

since

$$x^3=\frac{x}{2}(3x^2-\lambda)-\frac{1}{2}(x^3-\lambda x),$$

$$\lambda x=x^3-(x^3-\lambda x),$$

$$\lambda^2=3x(\lambda x)-\lambda(3x^2-\lambda).$$

Let $g\in\mathscr{E}^p_{x,\lambda}$. Using division, one can write

$$g(x,\lambda)=a_1+a_2x+a_3\lambda+a_4x^2+u_1(x,\lambda)\,x\lambda+u_2(x,\lambda)\,\lambda^2+u_3(x,\lambda)x^3, \tag{4.2}$$

where $a_i\in\mathbb{R}$ and $u_i\in\mathscr{E}^{p-3}_{x,\lambda}$. From (4.1), we have

$$u_1(x,\lambda)x\lambda+u_2(x,\lambda)\lambda^2+u_3(x,\lambda)x^3\in\tilde{T}^{p-3}_{\mathscr{G}}(f). \tag{4.3}$$

Since we also have $3x^2-\lambda\in\tilde{T}^{p-3}_{\mathscr{G}}(f)$, (4.2) and (4.3) imply that

$$\mathscr{E}^p_{x,\lambda}\subset\tilde{T}^{p-3}_{\mathscr{G}}(f)+\mathbb{R}\{1,x,x^2\}.$$

Now $x\in\mathscr{E}^\infty_\lambda\{\partial f/\partial\lambda\}$. Let $F(x,\lambda,\alpha_1,\alpha_2)=x^3-\lambda x+\alpha_1+\alpha_2x^2$. Then the assumptions of Theorem 4.2 are satisfied if we take $q=\infty$, $\sigma(p)=p-3$, $p_1=4$, $c=1$, $k=2$.

Proof of Theorem 4.2. For simplicity of notation we shall drop the superscript and subscript from $\tilde{\phi}$. Let $G(x,\lambda,\beta)$ be a C^{p+1} unfolding of f with $p_2\leqslant p\leqslant q$. We must show that G $(\mathscr{G},\tilde{\phi}(p))$-factors through F.

Let $H(x,\lambda,\alpha,\beta)=F(x,\lambda,\alpha)+G(x,\lambda,\beta)-f(x,\lambda)$, a C^{p+1} unfolding of f with $k+l$ parameters. Since $H(x,\lambda,0,\beta)=G(x,\lambda,\beta)$, G $(\mathscr{G},\infty)$-factors through H. We shall show that $H(x,\lambda,\alpha,\beta)$ $(\mathscr{G},\tilde{\phi}(p))$-factors through $H(x,\lambda,\alpha,\beta_1,...,\beta_{l-1},0)$. Since $H(x,\lambda,\alpha,0)=F(x,\lambda,\alpha)$, iterating the argument l times yields the result.

We further simplify the notation by setting $\delta=(\delta_1,...,\delta_{k+l-1})=(\alpha_1,...,\alpha_k,\beta_1,...,\beta_{l-1})$ and $\varepsilon=\beta_l$. Then we must show that $H(x,\lambda,\delta,\varepsilon)$

factors through $H(x, \lambda, \delta, 0)$. Note that every $K \in (\mathscr{E}^p_{x,\lambda,\delta,\varepsilon})^m$ can be written as

$$K(x, \lambda, 0, 0) + \sum_{i=1}^{k+l-1} K_i(x, \lambda, \delta, \varepsilon)\delta_i + K_{k+l}(x, \lambda, \delta, \varepsilon)\varepsilon,$$

where each $K_i \in (\mathscr{E}^{p-1}_{x,\lambda,\delta,\varepsilon})^m$. Then the main assumption of Theorem 4.2 implies

$$\begin{aligned}(\mathscr{E}^p_{x,\lambda,\delta,\varepsilon})^m \subset \mathscr{S}^{\sigma(p)}_{x,\lambda,\delta,\varepsilon}\{H\} + \mathscr{E}^{\sigma(p)}_{x,\lambda,\delta,\varepsilon}\left\{\frac{\partial H}{\partial x_1}, ..., \frac{\partial H}{\partial x_n}\right\} \\ + (\mathscr{E}^{p-1}_{x,\lambda,\delta,\varepsilon}\{\delta_1, ..., \delta_{k+l-1}, \varepsilon\})^m \\ + \mathbb{R}\left\{h_1, ..., h_c, \frac{\partial H}{\partial \delta_1}, ..., \frac{\partial H}{\partial \delta_k}\right\}.\end{aligned}$$

By the C^p Malgrange Preparation Theorem,

$$\begin{aligned}(\mathscr{E}^p_{x,\lambda,\delta,\varepsilon})^m \subset \mathscr{S}^{\tilde{\phi}(p)}_{x,\lambda,\delta,\varepsilon}\{H\} + \mathscr{E}^{\tilde{\phi}(p)}_{x,\lambda,\delta,\varepsilon}\left\{\frac{\partial H}{\partial x_1}, ..., \frac{\partial H}{\partial x_n}\right\} \\ + \mathscr{E}^{\tilde{\psi}(p)}_{\delta,\varepsilon}\left\{h_1, ..., h_c, \frac{\partial H}{\partial \delta_1}, ..., \frac{\partial H}{\partial \delta_k}\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}(\mathscr{E}^p_{x,\lambda,\delta,\varepsilon})^m \subset \mathscr{S}^{\tilde{\phi}(p)}_{x,\lambda,\delta,\varepsilon}\{H\} + \mathscr{E}^{\tilde{\phi}(p)}_{x,\lambda,\delta,\varepsilon}\left\{\frac{\partial H}{\partial x_1}, ..., \frac{\partial H}{\partial x_n}\right\} \\ + \mathscr{E}^{\tilde{\psi}(p)}_{\lambda,\delta,\varepsilon}\left\{\frac{\partial H}{\partial \lambda}\right\} + \mathscr{E}^{\tilde{\psi}(p)}_{\delta,\varepsilon}\left\{\frac{\partial H}{\partial \delta_1}, ..., \frac{\partial H}{\partial \delta_k}\right\}.\end{aligned} \tag{4.4}$$

We want to show that

$$H(x, \lambda, \delta, 0) = S(x, \lambda, \delta, \varepsilon)\, H(X(x, \lambda, \delta, \varepsilon), \Lambda(\lambda, \delta, \varepsilon), D(\delta, \varepsilon), \varepsilon) \tag{4.5}$$

with

$$S(x, \lambda, \delta, 0) = I, \tag{4.6}$$

$$S(x, \lambda, \delta, \varepsilon) \in \mathscr{G}, \tag{4.7}$$

$$X(x, \lambda, \delta, 0) = x, \tag{4.8}$$

$$\Lambda(\lambda, \delta, 0) = \lambda, \tag{4.9}$$

$$D(\delta, 0) = \delta, \tag{4.10}$$

$$S, X, \Lambda, D \text{ of class } C^{\tilde{\phi}(p)}. \tag{4.11}$$

Let a dot denote partial derivative with respect to ε. Then S, X, Λ, D must satisfy (4.6)–(4.10) and in addition

$$0 = \dot{S}H(X, \Lambda, D, \varepsilon) + S\{D_x H(X, \Lambda, D, \varepsilon)\dot{X} \\ + D_\lambda H(X, \Lambda, D, \varepsilon)\,\dot{\Lambda} + D_\delta H(X, \Lambda, D, \varepsilon)\,\dot{D} + D_\varepsilon H(X, \Lambda, D, \varepsilon)\}. \quad (4.12)$$

By (4.4) there exist $u \in \mathscr{S}^{\tilde{\phi}(p)}_{x,\lambda,\delta,\varepsilon}$, $v \in (\mathscr{E}^{\tilde{\phi}(p)}_{x,\lambda,\delta,\varepsilon})$, $w \in \mathscr{E}^{\tilde{\psi}(p)}_{\lambda,\delta,\varepsilon}$, $z \in (\mathscr{E}^{\tilde{\psi}(p)}_{\delta,\varepsilon})^{k+l-1}$ such that

$$-D_\varepsilon H(x, \lambda, \delta, \varepsilon) \\ = u(x, \lambda, \delta, \varepsilon)\, H(x, \lambda, \delta, \varepsilon) + D_x H(x, \lambda, \delta, \varepsilon)\, v(x, \lambda, \delta, \varepsilon) \\ + D_\lambda H(x, \lambda, \delta, \varepsilon)\, w(\lambda, \delta, \varepsilon) + D_\delta H(x, \lambda, \delta, \varepsilon)\, z(\delta, \varepsilon). \quad (4.13)$$

Successively define D, Λ, X, S by

$$\dot{D} = z(D(\delta, \varepsilon), \varepsilon), \qquad D(\delta, 0) = \delta.$$
$$\dot{\Lambda} = w(\Lambda(\lambda, \delta, \varepsilon), D(\delta, \varepsilon), \varepsilon), \qquad \Lambda(\lambda, \delta, 0) = \lambda.$$
$$\dot{X} = v(X(x, \lambda, \delta, \varepsilon), \Lambda(\lambda, \delta, \varepsilon), D(\delta, \varepsilon), \varepsilon), \qquad X(x, \lambda, \delta, 0) = x.$$
$$\dot{S} = S(x, \lambda, \delta, \varepsilon)\, u(X(x, \lambda, \delta, \varepsilon), \Lambda(\lambda, \delta, \varepsilon), D(\delta, \varepsilon), \varepsilon), \qquad S(x, \lambda, \delta, 0) = I.$$

Then D, Λ, X, S satisfy (4.6–4.10) ((4.7) follows from Lemma 2.3). By Lemma 4.5 at the end of this section, which says that $\tilde{\phi}(p) \leqslant \tilde{\psi}(p)$, they satisfy (4.11). By substituting the expressions for $\dot{D}, \dot{\Lambda}, \dot{X}, \dot{S}$ into the right-hand side of (4.12) and using (4.13) evaluated at $(X, \Lambda, D, \varepsilon)$, we see that (4.12) is satisfied. ∎

COROLLARY 4.4. *Let $f \in (\mathscr{E}^\infty_{x,\lambda})^m$ and let $F(x, \lambda, \alpha)$ be a C^∞ unfolding of f. Assume* (i) $((\mathscr{M}^\infty_{x,\lambda})^{(r+1)})^m \subset \tilde{T}^\infty_{\mathscr{G}}(f)$, *and* (ii) *there exist $h_1, \ldots, h_c \in \mathscr{E}^\infty_\lambda\{\partial f/\partial\lambda\}$ such that*

$$(\mathscr{E}^\infty_{x,\lambda})^m \subset \tilde{T}^\infty_{\mathscr{G}}(f) + \mathbb{R}\left\{h_1, \ldots, h_c, \frac{\partial F}{\partial \alpha_1}(x, \lambda, 0), \ldots, \frac{\partial F}{\partial \alpha_k}(x, \lambda, 0)\right\}.$$

Then (1) *F is $(\mathscr{G}, \infty, \infty)$-versal.*

(2) *Let $\sigma(p) = p - r - 1$. Let $p_2 = \min\{p : \tilde{\phi}^{(m, c+k; \sigma)}_{n+1}(p) > 0\}$. Then for all p such that $p_2 \leqslant p$, F is $(\mathscr{G}, \tilde{\phi}^{(m, c+k; \sigma)}_{n+1}(p), p+1)$-versal.*

Proof. The first conclusion follows from standard C^∞ singularity theory. To prove the second conclusion, we show that the assumptions of Theorem 4.2, with $q = \infty$ and $p_1 = r + 2$, are satisfied. For $p \geqslant r + 2$, let $g \in (\mathscr{E}^p_{x,\lambda})^m$. By Taylor's Theorem,

$$g = j^r(g) + \text{an element of } ((\mathscr{M}^{p-r-1}_{x,\lambda})^{(r+1)})^m. \quad (4.14)$$

Assumption (i) implies that

$$((\mathscr{M}_{x,\lambda}^{p-r-1})^{(r+1)})^m \subset \tilde{T}_{\mathscr{G}}^{p-r-1}(f). \tag{4.15}$$

By (ii),

$$j^r(g) = \sum_{i=1}^{c} a_i h_i + \sum_{j=1}^{k} b_j \frac{\partial F}{\partial \alpha_j}(x, \lambda, 0) + \tilde{g}, \tag{4.16}$$

where $a_i \in \mathbb{R}$, $b_j \in \mathbb{R}$, $\tilde{g} \in \tilde{T}_{\mathscr{G}}^{\infty}(f) \subset \tilde{T}_{\mathscr{G}}^{p-r-1}(f)$. Therefore (4.14–4.16) imply that

$$g = \sum_{i=1}^{c} a_i h_i + \sum_{j=1}^{k} b_j \frac{\partial F}{\partial \alpha_j}(x, \lambda, 0) + \hat{g}, \qquad \hat{g} \in \tilde{T}_{\mathscr{G}}^{p-r-1}(f). \quad \blacksquare$$

We now explain the functions $\tilde{\phi}_n$ and $\tilde{\psi}_n$ that appear in the statement of Theorem 4.1, and the relation of Theorem 4.1 to [8].

Let [ ] denote the greatest integer function. Given positive integers m and b, and d-functions σ and τ, Vegter defines new d-functions $\phi_n^{(m,b;\sigma,\tau)}(p)$ and $\psi_n^{(m,b;\tau)}(p)$, $n = 1, 2, ...,$ as follows:

$$\psi_1^{(m,b;\tau)}(p) = \left[\frac{\tau(p)+1}{b+m}\right] - 1;$$

$$\psi_n^{(m,b;\tau)}(p) = \psi_{n-1}^{(b,b;\tau)}(\psi_1^{(m,b;\tau)}(p));$$

$$\phi_1^{(m,b;\sigma,\tau)}(p) = \min\left\{\left[\frac{\tau(p)+1}{b+m}\right] - 1, \tau(\tau(p)), \sigma(\tau(p))\right\}.$$

To define ϕ_n, $n \geqslant 2$, Vegter first defines auxiliary functions

$$\bar{\sigma}^{(m,b;\sigma,\tau)}(p) = \min\{\sigma(\phi_1^{(m,b;\sigma,\tau)}(\psi_1^{(m,b;\tau)}(p))), \phi_1^{(m,b;\sigma,\tau)}(\tau(\psi_1^{(m,b;\tau)}(p)))\},$$

$$\bar{\tau}^{(m,b;\tau)}(p) = \psi_1^{(m,b;\tau)}(\tau(\psi_1^{(m,b;\tau)}(p))).$$

Then

$$\phi_n^{(m,b;\sigma,\tau)}(p) = \phi_{n-1}^{(b,b;\bar{\sigma},\bar{\tau})}(\psi_1^{(m,b;\tau)}(p)).$$

LEMMA 4.5. *Let σ and τ be d-functions with $\sigma(p) \leqslant \tau(p)$ for all p. Then for all n and p, $\phi_n^{(m,b;\sigma,\tau)}(p) \leqslant \psi_n^{(m,b;\tau)}(p)$.*

Proof. We prove the lemma by induction. For $n = 1$, we have

$$\phi_1^{(m,b;\sigma,\tau)}(p) \leqslant \left[\frac{\tau(p)}{b+m}\right] - 1 \leqslant \psi_1^{(m,b;\tau)}(p).$$

Therefore

$$\bar{\sigma}^{(m,b;\sigma,\tau)}(p) \leqslant \phi_1^{(m,b;\sigma,\tau)}(\tau(\psi_1^{(m,b;\tau)}(p)))$$
$$\leqslant \psi_1^{(m,b;\tau)}(\tau(\psi_1^{(m,b;\tau)}(p))) = \bar{\tau}^{(m,b;\tau)}(p). \tag{4.17}$$

Suppose that whenever $\sigma(p) \leqslant \tau(p)$ for all p, we have

$$\phi_{n-1}^{(m,b;\sigma,\tau)}(p) \leqslant \psi_{n-1}^{(m,b;\tau)}(p) \qquad \text{for all } p.$$

Then using (4.17), we have for all p:

$$\phi_{n-1}^{(b,b;\bar{\sigma},\bar{\tau})}(p) \leqslant \psi_{n-1}^{(b,b;\bar{\tau})}(p) \leqslant \psi_{n-1}^{(b,b;\tau)}(p).$$

Therefore the definitions of ϕ_n and ψ_n imply that for all p,

$$\phi_n^{(m,b;\sigma,\tau)}(p) \leqslant \psi_n^{(m,b;\tau)}(p). \quad \blacksquare$$

Now set $\tau(p) = p - 1$ and define

$$\bar{\phi}_n^{(m,b;\sigma)}(p) = \phi_n^{(m,b;\sigma,\tau)}(p);$$
$$\bar{\psi}_n^{(m,b)}(p) = \psi_n^{(m,b;\tau)}(p).$$

Then Theorem 4.1 is an immediate consequence of [8, Lemma 3.n], provided one notes that by Lemma 4.5, $\bar{\phi}_n^{(m,b;\sigma)}(p) > 0$ implies $\bar{\psi}_n^{(m,b)}(p) > 0$.

5. Normal Forms for $\mathscr{D}$- and $\mathscr{U}$-Equivalence

We recall from Section 1 that $\mathscr{D}$ denotes the group of 2×2 diagonal matrices with positive diagonal terms, and $\mathscr{U}$ denotes the group of 2×2 upper triangular matrices with positive diagonal terms. We shall concentrate on the group $\mathscr{D}$; then $\mathscr{S}$ is the space of 2×2 diagonal matrices. We consider ($\mathscr{D}$, ∞)-equivalence on $(\mathscr{E}_{x,\lambda}^{\infty})^2$, where $x \in \mathbb{R}$. We shall study two normal forms of codimension 0, two of codimension 1, one of codimension 2, and one family of codimension 3 normal forms with a modal parameter. For each normal form we shall use Theorem 3.10, with $s = p = \infty$, to solve the recognition problem (R) for ($\mathscr{D}$, ∞)-equivalence on $(\mathscr{E}_{x,\lambda}^{\infty})^2$. One could then use Corollary 3.11 to show that the solution is also valid for ($\mathscr{D}$, s)-equivalence on $(\mathscr{E}_{x,\lambda}^{p})^2$ for certain s and p, but we shall not do this. Similarly, we shall use the first conclusion of Corollary 4.4 to find a ($\mathscr{D}$, ∞, ∞)-universal unfolding for each normal form. Again, one could then use the second conclusion of Corollary 4.4 to show that the same unfolding is ($\mathscr{D}$, s, p)-universal for certain s and p, but we shall not. (A *universal unfolding* is a versal unfolding with minimum number of unfolding parameters; the *codimension* of a normal form is the number of unfolding parameters in a universal unfolding.) At the end of this section we mention briefly how the results change if one uses $\mathscr{U}$-equivalence.

Let us give an overview of how the results of this section relate to heteroclinic bifurcation.

Let $\dot{z} = h(z, \lambda)$ be a one-parameter family of vector fields on $\mathbb{R}^2$ having, for $\lambda = 0$, a hyperbolic saddle p_0 and a semihyperbolic equilibrium q_0 connected by a heteroclinic orbit Γ that lies in the stable manifold of q_0. Associated with h is a map $g(x, \lambda) = (g_1(x, \lambda), g_2(x, \lambda))$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that zeros of g_2 correspond to equilibria near q_0, and common zeros of g_1 and g_2 correspond to heteroclinic bifurcations. We have $g_1(0, 0) = g_2(0, 0) = g_{2x}(0, 0) = 0$ and $g_{1x}(0, 0) < 0$. A small perturbation of h near $\Gamma \times \{0\}$ gives rise to a small perturbation of g. The perturbation has the property that $g_1(x, \lambda) = 0$ implies $g_{1x}(x, \lambda) < 0$. Thus we shall limit our attention to normal forms (f_1, f_2) for which either $f_1(0, 0) \neq 0$, or $f_1(0, 0) = 0$ and $f_{1x}(0, 0) \neq 0$.

Generic perturbations of g will exhibit the following bifurcations:

(1) Saddle-node bifurcation, no heteroclinic bifurcation: points (x, λ) where $g_1 \neq 0$, $g_2 = g_{2x} = 0$, $g_{2\lambda} \neq 0$, $g_{2xx} \neq 0$.

(2) Nondegenerate heteroclinic bifurcation: points (x, λ) where $g_1 = g_2 = 0$, $g_{1x} \neq 0$, $g_{2x} \neq 0$, $g_{1x} g_{2\lambda} - g_{1\lambda} g_{2x} \neq 0$.

These two generic bifurcations appeared in Fig. 1.1 and are the codimension 0 bifurcations discussed below. Generic perturbations g have the additional property that at most one bifurcation occurs at each λ.

Nongeneric perturbations exhibit at least one of the following features:

($\mathscr{B}$) Equilibrium bifurcation: there is a point (x, λ) where $g_2 = g_{2x} = g_{2\lambda} = 0$.

($\mathscr{H}$) Hysteresis: there is a point (x, λ) where $g_2 = g_{2x} = g_{2xx} = 0$.

($\mathscr{E}$) Equilibrium/heteroclinic bifurcation: there is a point (x, λ) where $g_1 = g_2 = g_{2x} = 0$.

($\mathscr{N}$) Degenerate heteroclinic bifurcation: there is a point (x, λ) where $g_1 = g_2 = g_{1x} g_{2\lambda} - g_{1\lambda} g_{2x} = 0$.

($\mathscr{D}$) Double equilibrium bifurcation: there are distinct points (x_1, λ) and (x_2, λ) at which $g_2 = g_{2x} = 0$.

($\mathscr{F}$) Simultaneous heteroclinic and equilibrium bifurcations: there is a point (x_1, λ) at which $g_1 = g_2 = 0$ and a distinct point (x_2, λ) at which $g_2 = g_{2x} = 0$.

Let $F(x, \lambda, \alpha)$ be a universal unfolding of a normal form $f(x, \lambda)$. In α-space there will be codimension one *transition varieties* $\mathscr{B}$, $\mathscr{H}$, etc., of α-values for which $F(x, \lambda, \alpha)$, α fixed, exhibits the corresponding degeneracy. (In order that all these sets be closed, we define $\mathscr{D}$ (resp. $\mathscr{F}$) to be the closure of the set of α for which the corresponding degeneracy occurs.) For a typical α on $\mathscr{B}$ (resp. $\mathscr{H}$), $F(x, \lambda, \alpha)$, α fixed, has, after a translation in (x, λ), the normal form $f_1(x, \lambda) = \pm 1$, $f_2(x, \lambda) = \pm x^2 \pm \lambda^2$ (resp. $f_1(x, \lambda) = \pm 1$, $f_2(x, \lambda) = \pm x^3 \pm \lambda$). For a typical α on $\mathscr{E}$ (resp. $\mathscr{N}$), $F(x, \lambda, \alpha)$ has the codimension one normal form 5.3 (resp. 5.4) discussed below. For the codimension 2 and 3 normal forms discussed below, we shall sketch transition varieties in the corresponding α-spaces.

Since we plan to use Theorem 3.10 and Corollary 4.4, we note that because a basis for $\mathscr{S}$ is

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

we have

$$RT^p_{\mathscr{D}}(f) = \mathscr{E}^p_{x,\lambda}\{(f_1, 0), (0, f_2), xf_x, \lambda f_x\},$$

$$\tilde{T}^p_{\mathscr{D}}(f) = \mathscr{E}^p_{x,\lambda}\{(f_1, 0), (0, f_2), f_x\}.$$

Also, we note that according to Example 3.8, a simple $(\mathscr{D}, \infty)$-intrinsic submodule of $(\mathscr{E}^\infty_{x,\lambda)})^2$ has the form

$$M^\infty = (\mathscr{I}_1, 0) \oplus (0, \mathscr{I}_2) = (\mathscr{I}_1, \mathscr{I}_2),$$

where $\mathscr{I}_1$ and $\mathscr{I}_2$ are ideals of the form (3.9) with $s = \infty$ and $(\mathscr{I}_1, \mathscr{I}_2) \subset \mathscr{M}^\infty_{x,\lambda} \cdot RT^\infty_{\mathscr{D}}(f)$.

In the following, ε, δ, and γ are ± 1. Also, to simplify the notation we let $\mathscr{E}$ stand for $\mathscr{E}^\infty_{x,\lambda}$, $\mathscr{M}$ for $\mathscr{M}^\infty_{x,\lambda}$, and $\mathscr{E}_\lambda$ for $\mathscr{E}^\infty_\lambda$.

Normal Form 5.1. Saddle-node bifurcation:

$$f_1(x, \lambda) = \varepsilon,$$

$$f_2(x, \lambda) = \delta x^2 + \gamma \lambda.$$

Figures 5.1 and 5.2 show the corresponding equilibrium/heteroclinic bifurcation diagrams for $\varepsilon = \pm 1$, $\delta = 1$, $\gamma = -1$.

(R) $g \sim^\infty_{\mathscr{D}} f$ if and only if $g_2(0, 0) = g_{2x}(0, 0) = 0$, sgn $g_1(0, 0) = \varepsilon$, sign $g_{2xx}(0, 0) = \delta$, sgn $g_{2\lambda}(0, 0) = \gamma$.

(U) The universal unfolding of f is itself (i.e., f has codimension 0).

The proofs are omitted. In order to enable the reader to make the connection to $(\mathscr{D}, s)$-equivalence in $(\mathscr{E}^p_{x,\lambda})^2$, we note that $\mathscr{M} \cdot RT^\infty_{\mathscr{D}}(f)$ contains the simple $(\mathscr{D}, \infty)$-intrinsic submodule $(\mathscr{M}, \lambda\mathscr{M} + \mathscr{M}^{(3)})$, which

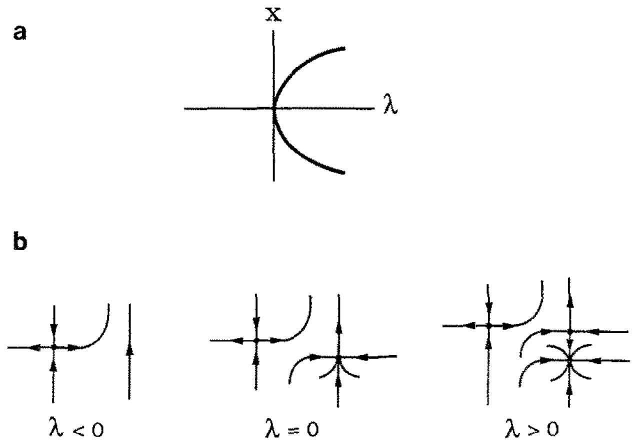

FIG. 5.1. $f(x, \lambda) = (1, x^2 - \lambda)$: (a) Bifurcation diagram; (b) phase portraits.

satisfies (i) of Theorem 3.10 with $r = 2$; and $\tilde{T}^{\infty}_{\mathscr{D}}(f) = (\mathscr{E}, \mathscr{M})$, which satisfies (i) of Corollary 4.4 with $r = 0$.

Normal Form 5.2. Nondegenerate heteroclinic bifurcation:

$$f_1(x, \lambda) = \varepsilon x,$$

$$f_2(x, \lambda) = \delta x + \gamma \lambda.$$

Figure 5.3 shows the corresponding heteroclinic bifurcation diagram for $\varepsilon = \gamma = -1$, $\delta = 1$, which involves only hyperbolic saddles. Notice that in

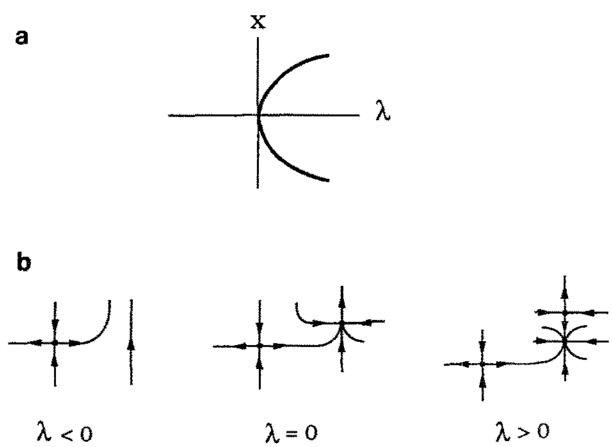

FIG 5.2. $f(x, \lambda) = (-1, x^2 - \lambda)$: (a) Bifurcation diagram; (b) phase portraits.

the bifurcation diagram of Fig. 5.3, and in the remaining bifurcation diagrams, the sets $f_1^{-1}(0)$ and $f_2^{-1}(0)$ are shown as dashed and solid curves, respectively. Notice also that for λ fixed, the relative x-coordinates of the dashed and solid curves allow one to read off the phase portrait from the bifurcation diagram.

(R) $g \sim_{\mathscr{D}}^{\infty} f$ if and only if $g_1(0,0)=g_2(0,0)=0$, $\operatorname{sgn} g_{1x}(0,0)=\varepsilon$, $\operatorname{sgn} g_{2x}(0,0)=\gamma$, and $\operatorname{sgn}(g_{1x}(0,0)\, g_{2\lambda}(0,0)-g_{1\lambda}(0,0)\, g_{2x}(0,0))=\varepsilon\gamma$.

(U) The universal unfolding of f is itself.

To prove (R), we note that

$$RT_{\mathscr{D}}^{\infty}(f)=\mathscr{E}\{(\varepsilon x, 0), (0, \delta x+\delta\lambda), (\varepsilon x, \delta x), (\varepsilon\lambda, \delta\lambda)\}.$$

Since $(x, 0)$, $(\lambda, 0)$, $(0, x)$, and $(0, \lambda)$ are all linear combinations of the generators of $RT_{\mathscr{D}}^{\infty}(f)$,

$$RT_{\mathscr{D}}^{\infty}(f) \supset (\mathscr{M}, \mathscr{M}).$$

Therefore

$$\mathscr{M} \cdot RT_{\mathscr{D}}^{\infty}(f) \supset (\mathscr{M}^{(2)}, \mathscr{M}^{(2)}).$$

Thus in Theorem 3.10 we let $M^{\infty}=(\mathscr{M}^{(2)}, \mathscr{M}^{(2)})$; then (i) is satisfied with $r=1$. Let $g \in \mathscr{E}^2$. We write

$$g_1(x, \lambda)=a_1+a_2 x+a_3\lambda+\cdots,$$

$$g_2(x, \lambda)=b_1+b_2 x+b_3\lambda+\cdots,$$

$$S(x, \lambda)=\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}+\cdots,$$

$$X(x, \lambda)=X_1 x+X_2\lambda+\cdots,$$

$$\Lambda(\lambda)=\Lambda_1\lambda+\cdots.$$

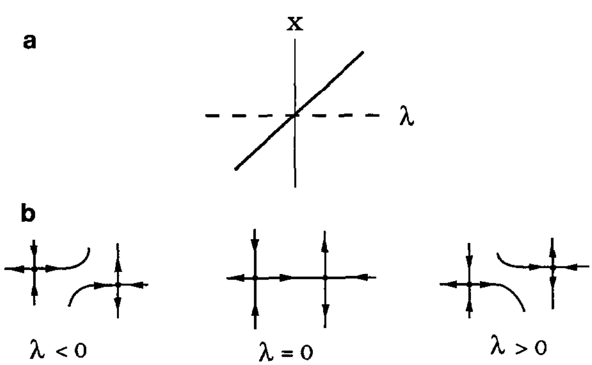

FIG. 5.3. $f(x, \lambda)=(-x, x-\lambda)$: (a) Bifurcation diagram; (b) phase portraits.

We then write the equation $g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda))$ modulo $(\mathcal{M}^{(2)}, \mathcal{M}^{(2)})$:

$$a_1 + a_2 x + a_3 \lambda = A_1 \cdot \varepsilon(X_1 x + X_2 \lambda),$$
$$b_1 + b_2 x + b_3 \lambda = B_1 \cdot \{\delta(X_1 x + X_2 \lambda) + \gamma \Lambda_1 \lambda\}.$$

One can find $A_1, B_1, X_1, X_2, \Lambda_1$, with all but possibly X_2 positive, satisfying these equations provided $a_1 = b_1 = 0$, sgn $a_2 = \varepsilon$, sgn $b_2 = \delta$, and

$$a_2 b_3 - a_3 b_2 = \varepsilon\gamma A_1 B_1 X_1 \Lambda_1.$$

This implies (R).

To prove (U) using the first conclusion of Corollary 4.4, we note that

$$\tilde{T}^{\infty}_{\mathscr{D}}(f) = \mathscr{E}\{(\varepsilon x, 0), (0, \delta x + \gamma\lambda), (\varepsilon, \delta)\} = (\mathcal{M}, \mathcal{M}) \oplus \mathbb{R}\{(1, 1)\}.$$

A complement is $\mathbb{R}\{(0, 1)\}$, and $(0, 1) \in \mathscr{E}_\lambda\{\partial f/\partial\lambda\}$. Therefore f has codimension 0.

Normal Form 5.3. Saddle-node/heteroclinic bifurcation:

$$f_1(x, \lambda) = \varepsilon x,$$
$$f_2(x, \lambda) = \delta x^2 + \gamma\lambda.$$

(R) $g \sim^{\infty}_{\mathscr{D}} f$ if and only if $g_1(0, 0) = g_2(0, 0) = g_{2x}(0, 0) = 0$, sgn $g_{1x}(0, 0) = \varepsilon$, sgn $g_{2xx}(0, 0) = \delta$, and sgn $g_{2\lambda}(0, 0) = \gamma$.

(U) A universal unfolding of f is

$$F(x, \lambda, \alpha) = (\varepsilon x + \alpha, \delta x^2 + \gamma\lambda).$$

The corresponding equilibrium/heteroclinic bifurcation diagrams for $\varepsilon = \gamma = -1$, $\delta = 1$ are shown in Fig. 5.4.

To prove (R), we note that

$$RT^{\infty}_{\mathscr{D}}(f) = \mathscr{E}\{(\varepsilon x, 0), (0, \delta x^2 + \gamma\lambda), (\varepsilon x, 2\delta x^2), (\varepsilon\lambda, 2\delta x\lambda)\}.$$

Since $(0, x^2)$ and $(0, \lambda)$ are linear combinations of the generators, $RT^{\infty}_{\mathscr{D}}(f)$ contains $(0, \lambda\mathscr{E} + \mathcal{M}^{(2)})$. Then from the fourth generator we see that $(\lambda, 0) \in RT^{\infty}_{\mathscr{D}}(f)$. Since $(x, 0) \in RT^{\infty}_{\mathscr{D}}(f)$ also,

$$RT^{\infty}_{\mathscr{D}}(f) \supset (\mathcal{M}, \lambda\mathscr{E} + \mathcal{M}^{(2)}).$$

Therefore

$$\mathcal{M} \cdot RT^{\infty}_{\mathscr{D}}(f) \supset (\mathcal{M}^{(2)}, \lambda\mathcal{M} + \mathcal{M}^{(3)}).$$

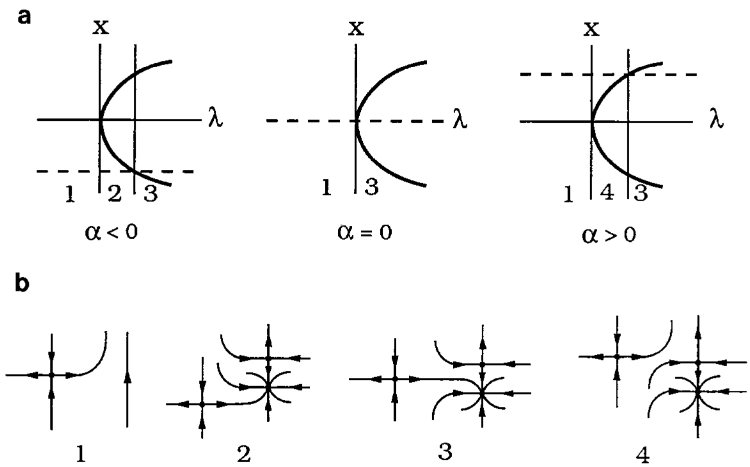

FIG. 5.4. $F(x, \lambda, \alpha) = (-x + \alpha, x^2 - \lambda)$: (a) Bifurcation diagrams; (b) phase portraits.

We write the equation $g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda))$ modulo $(\mathscr{M}^{(2)}, \lambda\mathscr{M} + \mathscr{M}^{(3)})$:

$$a_1 + a_2 x + a_3 \lambda = A_1 \cdot \varepsilon(X_1 x + X_2 \lambda),$$
$$b_1 + b_2 x + b_3 \lambda + b_4 x^2 = B_1 \cdot \{\delta X_1^2 x^2 + \gamma \Lambda_1 \lambda\}.$$

One can find $A_1, B_1, X_1, X_2, \Lambda_1$, with all but possibly X_2 positive, satisfying these equations provided $a_1 = b_1 = b_2 = 0$, $\operatorname{sgn} a_2 = \varepsilon$, $\operatorname{sgn} b_3 = \gamma$, $\operatorname{sgn} b_4 = \delta$. This implies (R).

To prove (U), we note that

$$\tilde{T}^{\infty}_{\mathscr{D}}(f) = \mathscr{E}\{(\varepsilon x, 0), (0, \delta x^2 + \gamma\lambda), (\varepsilon, 2\delta x)\}$$
$$= (\mathscr{M}, \lambda\mathscr{E} + \mathscr{M}^{(2)}) \oplus \mathbb{R}\{(\varepsilon, 2\delta x)\}.$$

A complement is $\mathbb{R}\{(1, 0), (0, 1)\}$. Since $\mathscr{E}_\lambda\{\partial f/\partial\lambda\} = \mathscr{E}_\lambda\{(0, 1)\}$, (U) follows.

Normal Form 5.4. Second-order heteroclinic bifurcation:

$$f_1(x, \lambda) = \varepsilon x,$$
$$f_2(x, \lambda) = \delta x + \gamma\lambda^2.$$

An easy necessary condition for g to be $(\mathscr{D}, \infty)$-equivalent to f is that $g_2(0, 0) = 0$ and $g_{2x}(0, 0) \neq 0$. For such g one can make a preliminary linear change of coordinates $x' = X(x, \lambda)$ after which $g_{2\lambda}(0, 0) = 0$. We state the solution of the recognition problem only for such g.

(R) If $g_2(0,0) = g_{2\lambda}(0,0) = 0$, then $g \sim_{\mathscr{D}}^{\infty} f$ if and only if $g_1(0,0) = g_{1\lambda}(0,0) = 0$, $\operatorname{sgn} g_{1x}(0,0) = \varepsilon$, $\operatorname{sgn} g_{2x}(0,0) = \delta$, $\operatorname{sgn}(g_{1x}(0,0)\, g_{2\lambda\lambda}(0,0) - g_{1\lambda\lambda}(0,0)\, g_{2x}(0,0)) = \varepsilon\gamma$.

(U) A universal unfolding of f is

$$F(x, \lambda, \alpha) = (\varepsilon x + \alpha, \delta x + \gamma\lambda^2).$$

In Fig. 5.5 we show the corresponding heteroclinic bifurcation diagrams for $\varepsilon = \gamma = -1$, $\delta = 1$; they involve only hyperbolic saddles.

To prove (R), we note that

$$RT_{\mathscr{D}}^{\infty}(f) = \mathscr{E}\{(\varepsilon x, 0), (0, \delta x + \gamma\lambda^2), (\varepsilon x, \delta x), (\varepsilon\lambda, \delta\lambda)\}.$$

Now $(x, 0)$, $(0, x)$, and $(0, \lambda^2)$ are linear combinations of the generators, and $(\varepsilon\lambda^2, 0)$ is λ times the fourth generator plus a multiple of $(0, \lambda^2)$. Therefore

$$RT_{\mathscr{D}}^{\infty}(f) \supset (\mathscr{M}^{(2)}, \mathscr{M}^{(2)}),$$

so

$$\mathscr{M} \cdot RT_{\mathscr{D}}^{\infty}(f) \supset (\mathscr{M}^{(3)}, \mathscr{M}^{(3)}).$$

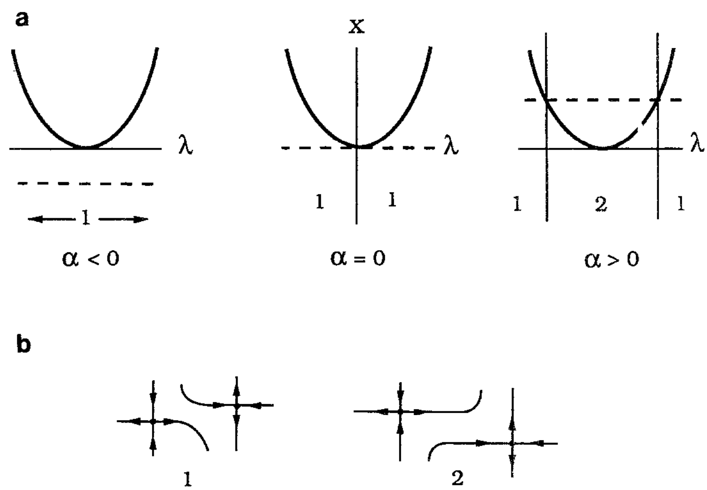

FIG. 5.5. $F(x, \lambda, \alpha) = (-x + \alpha, x - \lambda^2)$: (a) Bifurcation diagrams; (b) phase portraits.

We write the equation $g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda))$ modulo $(\mathscr{M}^{(3)}, \mathscr{M}^{(3)})$:

$$\begin{aligned}
&a_1 + a_2 x + a_3 \lambda + a_4 x^2 + a_5 x\lambda + a_6 \lambda^2 \\
&\quad = (A_1 + A_2 x + A_3 \lambda) \cdot \varepsilon(X_1 x + X_2 \lambda + X_3 x^2 + X_4 x\lambda + X_5 \lambda^2), \\
&0 + b_2 x + 0\lambda + b_4 x^2 + b_5 x\lambda + b_6 \lambda^2 \\
&\quad = (B_1 + B_2 x + B_3 \lambda) \\
&\qquad \cdot \{\delta(X_1 x + X_2 \lambda + X_3 x^2 + X_4 x\lambda + X_5 \lambda^2) + \gamma \Lambda_1^2 \lambda^2\}.
\end{aligned}$$

One can find A_i, B_i, X_i, and Λ_1, solving these equations modulo $(\mathscr{M}^{(3)}, \mathscr{M}^{(3)})$, with A_1, B_1, X_1, and Λ_1 positive, provided $a_1 = a_3 = 0$, $\operatorname{sgn} a_2 = \varepsilon$, $\operatorname{sgn} b_2 = \delta$, and $\operatorname{sign}(a_2 b_6 - a_6 b_2) = \varepsilon\gamma$. (In fact one may take $A_1 = 1$, $A_2 = A_3 = 0$.) This implies (R).

To prove (U), we note that

$$\begin{aligned}
\tilde{T}^{\infty}_{\mathscr{D}}(f) &= \mathscr{E}\{(\varepsilon x, 0), (0, \delta x + \gamma\lambda^2), (\varepsilon, \delta)\} \\
&= (x\mathscr{E} + \mathscr{M}^{(2)}, x\mathscr{E} + \mathscr{M}^{(2)}) \oplus \mathbb{R}\{(1, 1), (\lambda, \lambda)\}.
\end{aligned}$$

A complement is $\mathbb{R}\{(1, 0), (0, \lambda)\}$. Since $\mathscr{E}_\lambda\{\partial f/\partial\lambda\} = \mathscr{E}_\lambda\{(0, \lambda)\}$, (U) is a universal unfolding.

Normal Form 5.5. Hysteresis/heteroclinic bifurcation:

$$\begin{aligned}
f_1(x, \lambda) &= \varepsilon x, \\
f_2(x, \lambda) &= \delta x^3 + \gamma\lambda.
\end{aligned}$$

(R) $g \sim^{\infty}_{\mathscr{D}} f$ if and only if $g_1(0, 0) = g_2(0, 0) = g_{2x}(0, 0) = g_{2xx}(0, 0) = 0$, sgn $g_{1x}(0, 0) = \varepsilon$, sgn $g_{2\lambda}(0, 0) = \gamma$, sgn $g_{2xxx}(0, 0) = \delta$.

(U) A universal unfolding of f is

$$F(x, \lambda, \alpha_1, \alpha_2) = (\varepsilon x + \alpha_1, \delta x^3 + \gamma\lambda + \alpha_2 x).$$

At the top of Figure 5.6 we show, for $\varepsilon = \gamma = -1$, $\delta = 1$, transition curves in the α-plane: $\mathscr{H}(\alpha_2 = 0)$, $\mathscr{F}(\alpha_2 = -\frac{3}{4}\alpha_1^2)$, and $\mathscr{E}(\alpha_2 = -3\alpha_1^2)$. The curves divide the α-plane into open regions in which different stable bifurcation diagrams occur. These diagrams are also shown in Fig. 5.6.

To prove (R), we note that

$$RT^{\infty}_{\mathscr{D}}(f) = \mathscr{E}\{(\varepsilon x, 0), (0, \delta x^3 + \gamma\lambda), (\varepsilon x, 3\delta x^3), (\varepsilon\lambda, 3\delta x^2\lambda)\}.$$

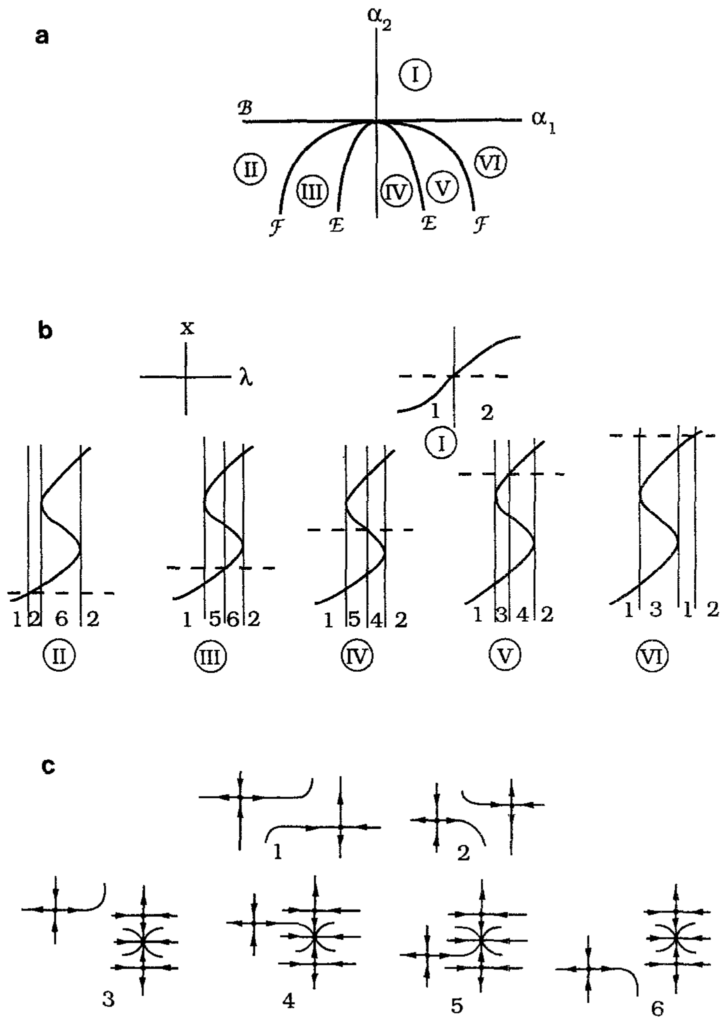

FIG. 5.6. $F(x, \lambda, \alpha_1, \alpha_2) = (-x + \alpha_1, x^3 - \lambda + \alpha_2 x)$: (a) Transition varieties; (b) bifurcation diagrams; (c) phase portraits.

Since $(x, 0)$, $(0, x^3)$, and $(0, \lambda)$ are linear combinations of the generators, and $(\varepsilon\lambda, 0)$ is the fourth generator plus a function times $(0, \lambda)$,

$$RT^{\infty}_{\mathscr{D}}(f) \supset (\mathscr{M}, \lambda\mathscr{E} + \mathscr{M}^{(3)}).$$

Therefore

$$\mathscr{M} \cdot RT^{\infty}_{\mathscr{D}}(f) \supset (\mathscr{M}^{(2)}, \lambda\mathscr{M} + \mathscr{M}^{(4)}).$$

We write the equation $g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda))$ modulo $(\mathcal{M}^{(2)}, \lambda\mathcal{M} + \mathcal{M}^{(4)})$:

$$a_1 + a_2 x + a_3 \lambda = A_1 \cdot \varepsilon(X_1 x + X_2 \lambda),$$
$$b_1 + b_2 x + b_3 \lambda + b_4 x^2 + b_5 x^3 = B_1 \cdot \{\delta X_1^3 x^3 + \gamma \Lambda_1 \lambda\}.$$

One can find $A_1, B_1, X_1, X_2, \Lambda_1$ satisfying these equations, with all but possibly X_2 positive, if and only if $a_1 = b_1 = b_2 = b_4 = 0$, $\operatorname{sgn} a_2 = \varepsilon$, $\operatorname{sgn} b_3 = \gamma$, $\operatorname{sgn} b_5 = \delta$. This implies (R).

To prove (U), we note that

$$\begin{aligned} \tilde{T}^{\infty}_{\mathscr{D}}(f) &= \mathscr{E}\{(\varepsilon x, 0), (0, \delta x^3 + \gamma\lambda), (\varepsilon, 3\delta x^2)\} \\ &= (\mathcal{M}, \lambda\mathscr{E} + \mathcal{M}^{(3)}) \oplus \mathbb{R}\{(\varepsilon, 3\delta x^2)\}. \end{aligned}$$

A complement is $\mathbb{R}\{(1, 0), (0, 1), (0, x)\}$. Since $\mathscr{E}_\lambda\{\partial f/\partial\lambda\} = \mathscr{E}_\lambda\{(0, 1)\}$, we have (U).

Normal Form 5.6. Transcritical/heteroclinic bifurcation:

$$f_1(x, \lambda) = \varepsilon(x + m\lambda), \ m \neq \pm 1,$$
$$f_2(x, \lambda) = \delta(x^2 - \lambda^2).$$

The parameter m is called a *modal* parameter; this term is used when there is a parameterized family of inequivalent normal forms of the same codimension. The fact that the normal forms (f_1, f_2) with distinct m are not smoothly equivalent is related to the cross-ratio invariant for collections of four lines through the origin in $\mathbb{R}^2$ under linear transformations. Here the four lines are $\lambda = 0$, $x + m\lambda = 0$, and $x \pm \lambda = 0$.

An easy necessary condition for g to be $(\mathscr{D}, \infty)$-equivalent to f is that $g_2(x, \lambda) = ax^2 + bx\lambda + c\lambda^2 + \cdots$ with $a \neq 0$ and $b^2 - 4ac \neq 0$. For such g one can make a preliminary linear change of coordinates $x' = X(x, \lambda)$ after which $g_2(x, \lambda) = b_1(x^2 - \lambda^2) + \cdots$. We state the solution of the recognition problem only for such g.

(R) If $g_2(x, \lambda) = b_1(x^2 - \lambda^2) + \cdots$, then $g \sim^{\infty}_{\mathscr{D}} f$ if and only if $\operatorname{sgn} b_1 = \delta$, $g_1(0, 0) = 0$, $\operatorname{sgn} g_{1x}(0, 0) = \varepsilon$, and $g_{1\lambda}(0, 0) = m g_{1x}(0, 0)$.

(U) A universal unfolding of f is

$$F(x, \lambda, \alpha_1, \alpha_2, \alpha_3) = (\varepsilon(x + (m + \alpha_1)\lambda) + \alpha_2, \delta(x^2 - \lambda^2) + \alpha_3).$$

Here m is regarded as fixed and α_1 is a modal parameter.

Let us discuss this universal unfolding for the case $\varepsilon = -1$, $\delta = 1$, $m = 0$. There are three transition varieties in α-space: $\mathscr{B}$ $(\alpha_3 = 0)$, $\mathscr{E}$ $(\alpha_1^2 \alpha_3 = \alpha_2^2, \alpha_3 \geqslant 0)$, and $\mathscr{N}$ $(\alpha_3 = \alpha_2^2/(\alpha_1^2 - 1))$. The variety $\mathscr{E}$ is Whitney's umbrella with the handle deleted, which is singular along the nonnegative α_3-axis. Points on this axis other than $\alpha = 0$ correspond to bifurcation diagrams in which *two* equilibrium/heteroclinic bifurcations occur.

In Fig. 5.7 we indicate how the transition varieties meet planes $\alpha_1 =$ constant, and we draw the stable bifurcation diagrams corresponding to the different open regions in the complement of these surfaces. One sees clearly from these pictures (or from the structure of the surface $\mathscr{E}$) that the normal form 5.6 with $m = 0$ is special. If $0 < |m| < 1$, the universal unfolding of 5.6 with α near 0 would not include any α-values corresponding to two equilibrium/heteroclinic bifurcations.

To prove (R), we note that

$$RT_{\mathscr{D}}^{\infty}(f) = \mathscr{E}\{(x + m\lambda, 0), (0, x^2 - \lambda^2), (\varepsilon x, 2\delta x^2), (\varepsilon\lambda, 2\delta x\lambda)\}.$$

Note that $(0, x^2 - \lambda^2)$ and $(0, x^2 + mx\lambda)$ are linear combinations of the generators. Multiplying $x^2 - \lambda^2$ and $x^2 + mx\lambda$ by x and λ, we obtain four

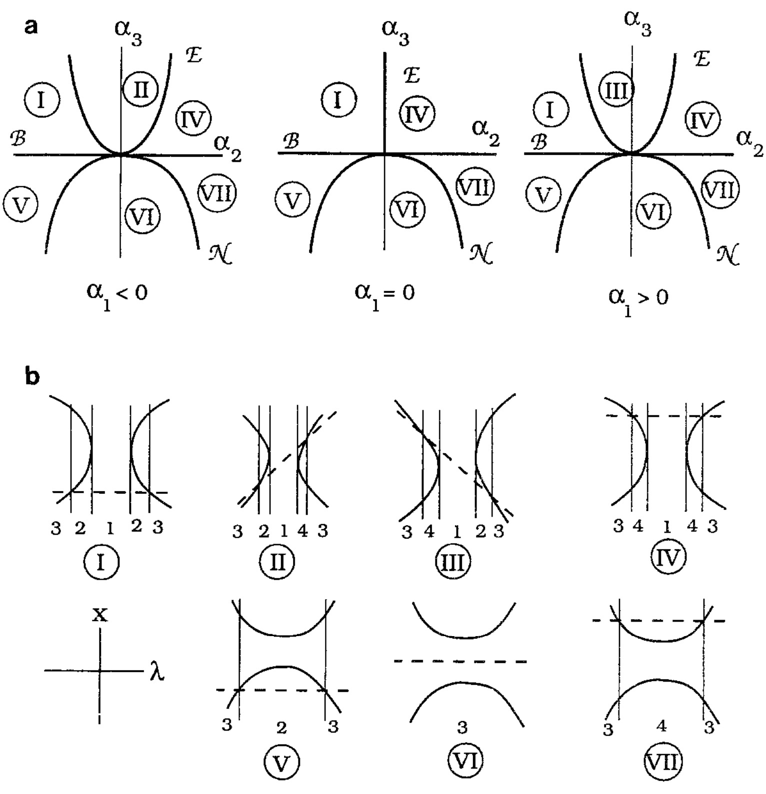

FIG. 5.7. $F(x, \lambda, \alpha_1, \alpha_2, \alpha_3) = (-(x + \alpha_1\lambda) + \alpha_2, x^2 - \lambda^2 + \alpha_3)$: (a) Transition varieties; (b) bifurcation diagrams. Phase portrait numbers refer to Fig. 5.4.

homogeneous cubic polynomials, which are linearly independent provided $m \neq \pm 1$. Therefore

$$RT^{\infty}_{\mathscr{D}}(f) \supset (0, \mathscr{M}^{(3)}).$$

Once this known, we use the last two generators to find that $RT^{\infty}_{\mathscr{D}}(f) \supset (\mathscr{M}^{(2)}, 0)$. Therefore

$$RT^{\infty}_{\mathscr{D}}(f) \supset (\mathscr{M}^{(2)}, \mathscr{M}^{(3)}),$$

so

$$\mathscr{M} \cdot RT^{\infty}_{\mathscr{D}}(f) \supset (\mathscr{M}^{(3)}, \mathscr{M}^{(4)}).$$

We write the equation $g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda))$ modulo $(\mathscr{M}^{(3)}, \mathscr{M}^{(4)})$:

$$\begin{aligned}
&a_1 + a_2 x + a_3 \lambda + a_4 x^2 + a_5 x\lambda + a_6 \lambda^2 \\
&\quad = (A_1 + A_2 x + A_3 \lambda) \\
&\qquad \cdot \varepsilon\{X_1 x + X_2 \lambda + X_3 x^2 + X_4 x\lambda + X_5 \lambda^2 + m(\Lambda_1 \lambda + \Lambda_2 \lambda^2)\}, \\
&b_1 x^2 - b_1 \lambda^2 + b_2 x^3 + b_3 x^2 \lambda + b_4 x\lambda^2 + b_5 \lambda^3 \\
&\quad = (B_1 + B_2 x + B_3 \lambda) \\
&\qquad \cdot \delta\{(X_1 x + X_2 \lambda + X_3 x^2 + X_4 x\lambda + X_5 \lambda^2)^2 - (\Lambda_1 \lambda + \Lambda_2 \lambda^2)^2\}.
\end{aligned}$$

One can find A_i, B_i, X_i, Λ_i satisfying these equations modulo $(\mathscr{M}^{(3)}, \mathscr{M}^{(4)})$, with A_1, B_1, X_1, Λ_1 positive, if and only if $a_1 = 0$, $\operatorname{sgn} a_2 = \varepsilon$, $\operatorname{sgn} b_1 = \delta$, and $a_3 = m a_2$. (In fact one can let $B_1 = 1$ and $A_2 = A_3 = B_2 = B_3 = \Lambda_2 = 0$.) This implies (R).

To show (U), we note that

$$\begin{aligned}
\tilde{T}^{\infty}_{\mathscr{D}}(f) &= \mathscr{E}\{(x + m\lambda, 0), (0, x^2 - \lambda^2), (\varepsilon, 2\delta x)\} \\
&= (\mathscr{M}^{(2)}, \mathscr{M}^{(3)}) \oplus \mathbb{R}\{z_1, ..., z_5\},
\end{aligned}$$

where $z_1 = (x + m\lambda, 0)$, $z_2 = (0, x^2 - \lambda^2)$, $z_3 = (\varepsilon, 2\delta x)$, $z_4 = (\varepsilon x, 2\delta x^2)$, $z_5 = (\varepsilon\lambda, 2\delta x\lambda)$. Also, $\mathscr{E}_\lambda\{\partial f/\partial\lambda\} = \mathscr{E}_\lambda\{(\varepsilon m, -2\delta\lambda)\}$, which contains $z_6 = (\varepsilon m, -2\delta\lambda)$ and $z_7 = (\varepsilon m\lambda, -2\delta\lambda^2)$. Now $z_1, ..., z_7$ lie in the obvious nine-dimensional complement to $(\mathscr{M}^{(2)}, \mathscr{M}^{(3)})$ in $\mathscr{E}^2$, spanned by monomials $(p, 0)$ and $(0, q)$ with $\deg p \leqslant 1$ and $\deg q \leqslant 2$. Moreover, $z_1, ..., z_6$ are linearly independent, and $z_7 = \varepsilon z_1 + 2\delta z_2 - z_4$. A basis for the complement is obtained by adding to $z_1, ..., z_6$ the vectors $(1, 0)$, $(\lambda, 0)$, and $(0, 1)$. This implies (U).

We end this section by briefly discussing $(\mathscr{U}, \infty)$-equivalence on $(\mathscr{E}^{\infty}_{x,\lambda})^2$.

According to Example 3.9 and Theorem 3.10, the recognition problem in $(\mathscr{E}^{\infty}_{x,\lambda})^2$ should be solved modulo a submodule of $(\mathscr{E}^{\infty}_{x,\lambda})^2$ of the form $(\mathscr{I}_1 + \mathscr{I}_2, \mathscr{I}_2)$, where $\mathscr{I}_1$ and $\mathscr{I}_2$ are ideals of the form (3.9) with $s = \infty$. One finds that in the solution of the recognition problem for normal forms 5.2 and 5.4, the condition $\operatorname{sgn} g_{1x}(0) = \varepsilon$ should be omitted. Otherwise the recognition criteria and universal unfoldings of normal forms 5.1–5.6 are unchanged.

References

1. J. Bochnak, M. Coste, and M.-F. Roy, "Géométrie Algébrique Réelle," Springer-Verlag, Berlin, 1987.
2. T. Gaffney, Some new results in the classification of bifurcation problems, *in* "Contemporary Mathematics," Vol. 56, pp. 97–118, American Mathematical Society, Providence, 1986.
3. M. Golubitsky and D. G. Schaeffer, "Singularities and Groups in Bifurcation Theory," Vol. 1, Springer-Verlag, Berlin, 1985.
4. M. Golubitsky, I. N. Stewart, and D. G. Schaeffer, "Singularities and Groups in Bifurcation Theory," Vol. 2, Springer-Verlag, Berlin, 1988.
5. S. Schecter, Simultaneous equilibrium and heteroclinic bifurcation of planar vector fields via the Melnikov integral, *Nonlinearity* **3** (1990), 79–99.
6. S. Schecter, Pitchfork bifurcation with a heteroclinic orbit: Normal form, recognition criteria, and universal unfolding, *J. Differential Equations*, to appear.
7. S. Schecter and M. Shearer, Undercompressive shocks for nonstrictly hyperbolic conservation laws, *J. Dynamics Differential Equations* **3** (1991), 199–271.
8. G. Vegter, "The C^p-preparation theorem, C^p-unfoldings and applications," Report ZW-8013, Mathematisch Instituut, Rijksuniversiteit, Groningen, 1980.
9. G. Vegter, Bifurcations of gradient vectorfields, *Astérisque* **98–99** (1982), 39–73.